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THE PRODUCT FORMULA FOR THE THIRD OBSTRUCTION

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Let ξ be an $SO(n)$-bundle with $n > 3$; let $p: E \to B$ be the projection in the associated $(n-1)$-sphere bundle. In this note we express the third obstruction to a cross-section of p as a tertiary characteristic class and prove a product formula for the behavior of this class under Whitney sum.

The first obstruction is the Euler class $\chi(\xi) \in H^n(B; Z)$. χ is a primary characteristic class and satisfies $\chi = j^*(U)$, where $j: B \to T$ is the inclusion into the Thom space and $U \in H^n(T; Z)$ is the Thom class. Whenever $\chi(\xi) = 0$, a secondary characteristic class

$$\alpha(\xi) \in H^{n+1}(B; Z_2)/(Sq^2 + w_2 \smile)H^{n-1}(B; Z)$$

is defined. α is the second obstruction and satisfies

$$\alpha = (Sq^2 + w_2 \smile)_j(U) .$$

Thus α is obtained by applying a twisted functional primary operation to U. The third obstruction $\gamma(\xi)$, defined whenever $\alpha(\xi) \equiv 0$, will be expressed as the value $\Phi_j(U)$ of a certain twisted functional secondary operation.

It is immediately plausible to consider as $(n+1)$-ary characteristic classes the values of certain functional twisted n-ary operations on U, defined when appropriate n-ary characteristic classes vanish. We hope to deal with such classes systematically in a future paper, but the treatment is expected to be more complicated technically; hence $\gamma(\xi)$ is presented here as an illustrative example in a straightforward setting.

The paper is organized as follows. Section 2 is a statement of results, while in § 3 we define $\gamma(\xi)$. The Peterson-Stein formula and the proof of (2.2) appears in § 4; the product formula is obtained in § 5. We conclude in § 6 with an example.

Throughout the paper all cohomology is taken with Z_2 as coefficients unless otherwise indicated.

2. Statement of results. Suppose ξ is an $SO(n)$-bundle with $n > 3$ and suppose $\chi(\xi) = 0$. Let

$$\alpha(\xi) \in H^{n+1}(B)/(Sq^2 + w_2 \smile)H^{n-1}(B; Z)$$

be the secondary characteristic class given by $\alpha(\xi) = (Sq^2 + w_2 \smile)_j(U)$

[5, 6, 7, 9]. By [9], $\alpha(\xi)$ is the second obstruction to a cross-section in the associated sphere bundle.

Suppose now $\alpha(\xi) \equiv 0$. Then in § 3 is defined a tertiary characteristic class $\gamma(\xi) \in H^{n+2}(B)$ modulo an indeterminacy Q, given in (3.6). γ is natural in the following sense.

PROPOSITION 2.1. $f: \xi' \rightarrow \xi$ be a map of $SO(n)$-bundles. Suppose $\gamma(\xi)$ is defined. Then $\gamma(\xi') \equiv f^*(\gamma(\xi)) \bmod Q(\xi')$.

In § 4 we establish the following.

PROPOSITION 2.2. $\gamma(\xi)$ is the third obstruction to a cross-section of p.

For product farmulas we now assume ξ and ξ' are $SO(n)$ and $SO(n')$-bundles over B and B' respectively such that $\alpha(\xi)$ and $\alpha(\xi')$ are defined. Let $\xi \oplus \xi'$ be the external Whitney sum over $B \times B'$. By the Whitney formula for secondary characteristic classes [9], $\alpha(\xi \oplus \xi') \equiv 0$ and thus $\gamma(\xi \oplus \xi')$ is defined. In § 5 we prove the following.

PROPOSITION 2.3. $\gamma(\xi \oplus \xi') \equiv \alpha(\xi) \otimes \alpha(\xi')$ modulo the total indeterminacy.

Taking $B = B'$ and writing $\xi + \xi'$ for the internal Whitney sum, we obtain the following corollary to (2.1) and (2.3).

PROPOSITION 2.4. $\gamma(\xi + \xi') \equiv \alpha(\xi) \smile \alpha(\xi')$ modulo the total indeterminacy.

3. Definition of $\gamma(\xi)$. Let A be the mod 2 Steenrod algebra. In the semi-tensor product $H^*(BSO) \odot A$ [3] we have, in the terminology of [11], the relation

$$(3.1) \qquad (1 \otimes Sq^2 + w_2 \otimes 1)(1 \otimes Sq^2 + w_2 \otimes 1) = 0$$

over Z. Let $\beta = 1 \otimes Sq^2 + w_2 \otimes 1$. According to [4] and [11], (3.1) defines for each n sufficiently large ($n > 2$ suffices in this case) a twisted secondary operation $\Phi^{(n)}$. $\Phi^{(n)}$ is defined on an n-dimensional integral cohomology class x of a space X, where $\beta x = 0$ and $H^*(BSO) \times A$ acts on the cohomology of X *via* a vector bundle. The indeterminacy of $\Phi^{(n)}(X)$ is the subgroup $\beta H^{n+1}(X)$ of $H^{n+3}(X)$. While $\Phi^{(n)}$ is not uniquely determined by (3.1), computation in the universal example verifies the following for $n > 2$.

PROPOSITION 3.2. For each n, there exist precisely two distinct operations $\Phi_1^{(n)}$ and $\Phi_2^{(n)}$ associated with (3.1); these operations are related by $\Phi_1^{(n)}(x) + \Phi_2^{(n)}(x) = Sq^3x = w_3 \smile x$.

Let U_n be the Thom class of the universal $SO(n)$-bundle γ_n. Another calculation checks the following.

PROPOSITION 3.3. *For each n, there is a unique choice of $\Phi^{(n)}$ such that $\Phi^{(n)}(U_n) = 0$.*

We now assume that $\Phi^{(n)}$ are so chosen and further note that $\Phi^{(n)}$ so chosen are compatible with coboundary, as is verified by consideration of the natural map $T(\gamma_{n-1} + 1) \to T(\gamma_n)$ of Thom spaces.

Suppose now the $SO(n)$-bundle ξ satisfies $\chi(\xi) = 0$ and $\alpha(\xi) \equiv 0$. Then U satisfies $j^*(U) = 0$, $\beta(U) = 0$, $\beta_j(U) \equiv 0$, and $\Phi(U) = 0$ with zero indeterminacy. Under these circumstances one defines $\Phi_j(U)$ by the analogue for twisted operations of Peterson's generalization [8] of Steenrod's basic method [10], detailed below; one then defines $\gamma(\xi)$ as follows.

DEFINITION 3.4. $\gamma(\xi) = \Phi_j(U)$.

To define $\Phi_j(U)$, following Massey [2], consider the cohomology sequence of the pair (B, E) where B replaces the mapping cylinder of p. Since $\chi(\xi) = j^*(U) = 0$, we may choose $a \in H^{n-1}(E; Z)$ such that $\delta^*(a) = U$. Since $\alpha(\xi) \equiv 0$, a may be further assumed to satisfy $\beta(a) = 0$. Then $\Phi(a)$ is defined and satisfies

$$\delta^*\Phi(a) = \Phi(\delta^*(a)) = \Phi(U) = 0 .$$

DEFINITION 3.5. $p^*(\Phi_j(U)) = \bigcup \Phi(a)$ *as a ranges over elements $a \in H^{n-1}(E; Z)$ such that $\delta^*(a) = U$ and $(Sq^2 + w_2 \smile)(a) = 0$.*

PROPOSITION 3.6. *The indeterminacy Q of $\gamma(\xi)$ is given by*

$$Q = \{\Phi(b) + \beta(c)\} ,$$

where $b \in H^{n-1}(B; Z)$ such that $\Phi(b)$ is defined and $c \in H^n(B)$.

(3.6) and (2.1) are now evident.

4. The Peterson-Stein formula and the proof of (2.2). Twisted secondary operations satisfy the usual Peterson-Stein formulas. Stated as (4.1), for simplicity in terms of absolute cohomology classes, is the one to be used.

PROPOSITION 4.1. Let $f\colon Y \to X$ be a map compatible with the given structures of Y and X as spaces obtained from vector bundles. Let $x \in H^n(X; Z)$ satisfy $\beta(f^*(x) = 0$. Then

$$\Phi(f^*(x)) \equiv \beta_f\beta(x) \in H^{n+3}(Y) \bmod \beta H^{n+1}(Y) + f^*H^{n+3}(X) .$$

The proof of (4.1) is postponed to the end of this section. The functional operation β_f appearing in (4.1) is defined by the generalization of Steenrod's method as given in [7].

We now turn to the proof of (2.2). Consider the portion of the Moore-Postinkov tower for the associated sphere bundle to the universal $SO(n)$-bundle γ_n displayed in (4.2).

Diagram 4.2.
$$\begin{array}{ccc} B_2 & \xrightarrow{k_2} & K(Z_2, n+2) \\ \downarrow{\scriptstyle q_1} & & \\ B_1 & \xrightarrow{k_2} & K(Z_2, n+1) \\ \downarrow{\scriptstyle q_1} & & \\ BSO(n) & \xrightarrow{\chi} & K(Z, n)\ . \end{array}$$

Let $\xi_1 = q_1^*(\gamma_n)$ and $\xi_2 = q_2^*(\xi_1)$. It then suffices to show $k_2 \in \gamma(\xi_2)$. By [9] $k_1 \in \alpha(\xi_1)$, while, by [1], $k_2 \in \beta_{q_2}(k_1)$.

Consider now (4.3), induced by the bundle map $q_2\colon \xi_2 \to \xi_1$.

Diagram 4.3.
$$\begin{array}{ccc} E_2 & \xrightarrow{p_2} & B_2 \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_2} \\ E_1 & \xrightarrow{p_1} & B_1 \end{array}$$

Since $k_1 \in \alpha(\xi_1)$, we may write $p_1^*(k_1) = \beta(a_1)$ for an appropriate $a_1 \in H^{n-1}(E_1)$ such that $\delta^*(a_1) = U(\xi_1)$. Let $a_2 = q_2^*(a_1)$. Then $(p_2^*)^{-1}\Phi(a_2)$ represents $\gamma(\xi_2)$.

On the other hand, since $k_2 \in \beta_{q_2}(k_1)$, by naturality

$$p_2^*(k_2) \in \beta_{q_2}(p_1^*(k_1)) = \beta_{q_2}\beta(a_1) .$$

The result follows by (4.1), which yields $\beta_{q_2}\beta(a_1) \equiv \Phi(a_2)$.

Proof of (4.1). For this proof we adopt the notations of [11]. Let $p\colon E, Y \to Y \times K, Y$ be the universal example for Φ. Then a representative φ of $\Phi(p^*(\iota_n))$ is defined in [11] by means of a certain relative transgression sequence for p by a formula $\varphi \in \mu^{-1}\alpha\tau^{-1}\beta(\iota_n)$. However, it is proved in [12] that this transgression sequence, in the range of dimensions considered, is equivalent to the cohomology sequence of the triple (M, E, Y), where M is the mapping cylinder of

p. Let $j\colon Y \times K, Y \to M, E$ be the inclusion. Translating the definition of φ to this sequence, we have $\varphi \in (\delta^*)^{-1}\beta(j^*)^{-1}\beta(\iota_n)$. But this last is precisely the definition of a representative of $\beta_p\beta(\iota_n)$. Thus (4.1) is valid in the universal example, and hence in general.

5. **Proof of (2.3).** We now consider bundles ξ and ξ' such that $\alpha(\xi)$ and $\alpha(\xi')$ are defined; let $\xi'' = \xi \oplus \xi'$. Denote by Z the mapping cylinder of p. The following is proved in [7].

PROPOSITION 5.1. *There is a natural homeomorphism of pairs Z'', $E'' \to Z \times Z', E \times Z' \smile Z \times E'$ extending the identity of $B'' = B \times B'$ and inducing a natural homeomorphism $T'' \to T'' \wedge T'$.*

Now consider (5.2), in which the rows and the middle triangle are exact. The top row of (5.2) is obtained by splicing $(0 \to H^*(B) \to H^*(E) \to H^*(T) \to 0) \otimes H^*(B')$ with $H^*(T) \otimes (0 \to H^*(B') \to H^*(E') \to H^*(T') \to 0)$, while the triangle is the exact sequence of the pair E'', $E \times Z'$.

Diagram 5.2.

$$\begin{tikzcd}
0 \arrow[r] & H^*(B'') \arrow[r, "p^* \otimes 1"] \arrow[d, equal] & H^*(E) \otimes H^*(B') \arrow[rr, dashed, "\delta^* \otimes p'^*"] & & H^*(T) \otimes H^*(E') \arrow[r, dashed, "1 \otimes \delta'^*"] \arrow[dl, dashed, "f^*"'] & H^*(T'') \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^*(B'') \arrow[rr, "p''^*"] & & H^*(E'') \arrow[ul, "g^*"'] \arrow[rr, "\delta''^*"] & & H^*(T'') \arrow[r] & 0
\end{tikzcd}$$

The proof of (2.3) is based on (5.2) as follows. Choose $a' \in H^{n'-1}(E')$ such that $\delta'^*(a') = U'$. Let $a'' = f^*(U \otimes a')$. Then $\delta''^*(a'') = U''$. Further, $(Sq^2 + w_2'' \smile)(a'') = 0$, as calculation checks. Thus $(p''^*)^{-1}\Phi(a'')$ is a representative of $\gamma(\xi + \xi')$.

On the other hand, $\Phi(a'') = \Phi(f^*(U \otimes a'))$ may be evaluated by (4.1). Computing, using the Wu formula [9] $(Sq^2 + w_2 \smile)(a) = 0$ and denoting by a any class in $H^{n-1}(E; Z)$ such that $\delta^*(a) = U$, we have the following, in which $\alpha(a)$ is the representative of α determined by a.

$$\begin{aligned}
(p''^*)^{-1}\Phi(a'') &= (p''^*)^{-1}\Phi(f^*(U \otimes a')) \\
&= (p''^*)^{-1}\beta''_f\beta''(U \otimes a') \\
&= (p''^*)^{-1}\beta''_f[U \otimes \beta'(a')] \\
&= (p''^*)^{-1}(g^*)^{-1}\beta''[a \otimes \alpha'(a')] \\
&= (p^* \otimes 1)^{-1}[\beta(a) \otimes \alpha'(a')] \\
&= \alpha(a) \otimes \alpha'(a')
\end{aligned}$$

modulo indeterminacies.

This completes the proof of (2.3) and in fact of the following sharpening.

COROLLARY 5.3. *Under the hypotheses of* (2.3), *let* $\alpha(a)$ *and* $\alpha'(a')$ *be representatives of* $\alpha(\xi)$ *and* $\alpha(\xi')$ *respectively. Then* $\alpha(a) \otimes \alpha'(a')$ *is a representative of* $\gamma(\xi \oplus \xi')$.

6. An example. Let $\xi + 1$ be the tangent bundle of S^{4q+1} and $\xi' + 1$ the tangent bundle of $S^{4q'+1}$ for $q, q' \geqq 1$. By [9], $\alpha(\xi) \neq 0 \bmod 0$ in $H^{4q+1}(S^{4q+1})$ and similarly for ξ'. It follows by (2.3) that $\gamma(\xi \oplus \xi')$ is nonzero in $H^{4q+4q'+2}(S^{4q+1} \times S^{4q'+1})$; the indeterminacy again vanishes. Thus $\xi \oplus \xi'$ has no nonvanishing section.

This result can be obtained without the use of twisted operations, for the Whitney classes here vanish. That $\alpha(\xi) \neq 0$ reflects that $Sq^2 a$ generates $p^* H^{4q+1}(S^{4q+1})$ in $H^{4q+1}(E)$, while $\gamma(\xi + \xi') \neq 0$ reflects that $\Phi_{1,1}(a'')$ generates $p''^* H^{4q+4q'+2}(S^{4q+1} \times S^{4q'+1})$ in $H^{4q+4q'+2}(E'')$, where $\Phi_{1,1}$ is the ordinary secondary operation associated with the Adem relation $Sq^2 Sq^2 = 0$, valid on integer classes.

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