PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 1, 1969

ALGEBRAS SATISFYING THE DESCENDING CHAIN CONDITION FOR SUBALGEBRAS

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In this paper we give a partial solution to the following problem of B. Jónsson:

(*) For which cardinals m do there exist algebras of power m having finitely many operations and satisfying the descending chain condition for subalgebras?

Of course a necessary condition for the existence of such an algebra is that there exist an algebra of power m having finitely many operations and having no proper subalgebra of power m. The first such construction was by F. Galvin who constructed an algebra of power ω_1 which satisfied the descending chain condition for subalgebras. It has been shown by Erdos and Hajnal [1] that for $n \in \omega$ there is an algebra of power ω_n which has finitely many operations and has no proper subalgebra of power ω_n. Actually C. C. Chang [3] has shown that if an algebra exists of power m having finitely many operations and having no proper subalgebra of power m, then such an algebra exists of power m^+. In § 2 we modify this construction to show that if there is an algebra of power m with finitely many operations and satisfying the descending chain condition, then there is such an algebra of power m^+.

Erdos and Hajnal [1] also showed, under the assumption of the generalized continuum hypothesis, that for any cardinal m there is a locally finite algebra of power m^+ having finitely many operations and having no proper subalgebra of power m^+. In § 3 we show that for $n \in \omega$ there is a locally finite algebra of power ω_n having finitely many operations and satisfying the descending chain condition for subalgebras.

2. General algebras.

Before beginning the construction of the algebras we note the following relevant theorem of W. Hanf.

THEOREM 2.1. (Hanf [2], [4]). *The lattice of subalgebras of an algebra with countably many operations is a compactly generated lattice in which each compact element contains at most countably many compact elements. Conversely, any such lattice can be realized as the lattice of subalgebras of a commutative loop in which each subalgebra is a subloop.*

COROLLARY 2.2. *The following are equivalent:*

(i) *There exists a compactly generated lattice having m compact elements in which each compact element contains at most countably* *many compact elements and which satisfies the descending chain condition (for elements).*

(ii) *There is an algebra of power m having countably many operations and satisfying the descending chain condition for subalgebras.*

(iii) *There is an algebra of power m having finitely many operations and satisfying the descending chain condition for subalgebras.*

(iv) *There is a commutative loop of power m satisfying the descending chain condition for subalgebras.*

THEOREM 2.3. *If there is an algebra of power m having finitely many operations and satisfying the descending chain condition for subalgebras, then there is an algebra of power m^+ having finitely many operations and satisfying the descending chain condition for subalgebras.*

Proof. Suppose we have such an algebra of power m. Using Corollary 2.2 we assume our algebra is of the form $A = \langle m; f \rangle$ (identifying the cardinal m with the set of all ordinals of cardinality less than m). Actually we could take A to be a commutative loop, but these properties are not needed here. For each ordinal ξ with $m \leqq \xi < m^+$, let ϕ_ξ be a one-to-one map of ξ onto m. We now define a binary operation $\bar{f}$ on m^+ by

$$\bar{f}(\eta_0, \eta_1) = \begin{cases} f(\eta_0, \eta_1) \text{ if } \eta_0, \eta_1 < m\ , \\ \phi_{\eta_0}(\eta_1) \text{ if } m \leqq \eta_0 \text{ and } \eta_1 < \eta_0\ , \\ \phi_{\eta_1}^{-1}(\eta_0) \text{ if } \eta_0 < m \leqq \eta_1\ , \\ 0 \text{ otherwise}\ . \end{cases}$$

We show that $A' = \langle m^+; \bar{f} \rangle$ has the desired properties.

If B is a subalgebra of A' ($B \subseteqq_s A'$) then it is clear that $B \cap m$ is a subalgebra of A. Furthermore, if $m \leqq \xi \in B$ we can see that $m \cap B = \phi_\xi(\xi \cap B)$. To see this note that if $\eta \in \xi \cap B$ then $\phi_\xi(\eta) = \bar{f}(\xi, \eta) \varepsilon\, m \cap B$ while if $\eta' \in m \cap B$ then $\phi_\xi^{-1}(\eta') = \bar{f}(\eta', \xi) \in \xi \cap B$.

We now show that if $C \subset_s B \subset_s A'$, one of the following three conditions must hold:

(i) $C \cap m \subset_s B \cap m$,

(ii) $\Sigma C < \Sigma B$,

(iii) $\Sigma B \in B - C$.

Assume that $\Sigma C = \Sigma B$ and $\Sigma B \notin B - C$. Suppose first that B has a largest member, β. Then $\beta = \Sigma B \notin B - C$ implying that $\beta \in C$. Thus $C \cap \beta \subset B \cap \beta$. We know that $C \cap m = \phi_\beta(C \cap \beta) \subset \phi_\beta(B \cap \beta) = B \cap m$. This leaves only the case where B has no largest member. Take $\xi \in B - C$. If $\xi < m$, we have $C \cap m \subset B \cap m$. Therefore we assume that $m \leqq \xi < m^+$. Since $\Sigma B = \Sigma C > \xi$, there is a $\xi' \in C$ with $\xi < \xi'$. Then $\xi' \cap C \subset \xi' \cap B$ so $m \cap C = \phi_{\xi'}(\xi' \cap C) \subset \phi_{\xi'}(\xi' \cap B) = m \cap B$.

Suppose we have $A' \supseteqq_s B_0 \supseteqq_s B_1 \supseteqq_s \cdots$. Clearly $\Sigma B_0 \geqq \Sigma B_1 \geqq \cdots$. There is some $k_0 \in \omega$ so that $\Sigma B_{k_0} = \Sigma B_{k_0+1} = \cdots$. Also we know that

$$A \supseteqq_s B_{k_0} \cap m \supseteqq_s B_{k_0+1} \cap m \supseteqq_s \cdots .$$

Since A satisfies the descending chain condition for subalgebras, there is a $k_1 \geqq k_0$ so that $B_{k_1} \cap m = B_{k_1+1} \cap m = \cdots$. Assume now that $n_1 < n_2 < \cdots$ and that $B_{k_1} \supset B_{k_1+n_1} \supset B_{k_1+n_2} \supset \cdots$. Of the three conditions listed above, only (iii) applies to $B_{k_1+n_2} \subset_s B_{k_1+n_1} \subset_s A'$. Thus $\Sigma B_{k_0} \in B_{k_1+n_1} - B_{k_1+n_2}$. Similarly, we get $\Sigma B_{k_0} \in B_{k_1+n_2} - B_{k_1+n_3}$. This contradiction completes the proof.

COROLLARY 2.4. *For $n \in \omega$ there is a commutative loop of power ω_n satisfying the descending chain condition for subalgebras.*

3. Locally finite algebras.

By a locally finite algebra we mean an algebra in which each finite subset generates a finite subalgebra. The following theorem characterizes the lattices of subalgebras of locally finite algebras in a manner somewhat analogous to Hanf's theorem.

THEOREM 3.1. *The lattice of subalgebras of a locally finite algebra is a compactly generated lattice in which each compact element contains only finitely many compact elements. Conversely, any such lattice may be realized as the lattice of subalgebras of a locally finite algebra having one commutative binary operation.*

Proof. Since the compact elements in the lattice of subalgebras of an algebra correspond to the finitely generated subalgebras and since each finitely generated subalgebra of a locally finite algebra is finite, it is clear that each compact element in the lattice of subalgebras of a locally finite algebra contains only finitely many compact elements.

Conversely, suppose $\langle L; +, \cdot \rangle$ is a compactly generated lattice in which each compact element contains only finitely many compact elements. Let L^c be the semilattice of compact elements of L. We know that L is isomorphic to the lattice of ideals of L^c. We now define a commutative binary operation, f, on L^c so that the subalgebras of $\langle L^c; f \rangle$ are precisely the ideals of $\langle L^c; + \rangle$ with the finitely generated subalgebras just the principal ideals. This will clearly complete the proof. For $a \in L^c$ let $\{a_0, a_1, \cdots, a_{n(a)}\}$ be the principal ideal of $\langle L^c; + \rangle$ generated by a with $a = a_0$ and $a_i \neq a_j$ if $i \neq j$. Define f by

$$f(a, b) = \begin{cases} a_{j+1} \text{ if } b = a_j \text{ with } j < n(a) \\ b_{j+1} \text{ if } a = b_j \text{ with } j < n(b) \\ a + b \text{ otherwise.} \end{cases}$$

It is easy to check that the subalgebras of $\langle L^c; f\rangle$ are as described above.

COROLLARY 3.2. *For any* m *the following are equivalent:*

(i) *There is a compactly generated lattice having* m *compact elements in which each compact element contains only finitely many compact elements and which satisfies the descending chain condition.*

(ii) *There is a locally finite algebra of power* m *which satisfies the descending chain condition for subalgebras.*

(iii) *There is a locally finite algebra of power* m *having one commutative binary operation and satisfying the descending chain condition for subalgebras.*

THEOREM 3.3. *For* $n \in \omega$ *there is a locally finite algebra of power* ω_n *which satisfies the descending chain condition for subalgebras.*

Proof. The proof will be by induction on n. First we construct A_0 of power ω. For each $m \in \omega$ define a unary operation $f_{m,0}$ on ω by

$$f_{m,0}(n) = \begin{cases} n - m \text{ if } m \leqq n\,, \\ 0 \text{ otherwise}\,. \end{cases}$$

We then let $A_0 = \langle \omega; f_{m,0}\rangle_{m \in \omega}$.

As an induction hypothesis we assume that we have

$$A_n = \langle \omega_n; f_{m,n}, \omega_s\rangle_{\substack{m \in \omega \\ s < n}}$$

so that the following assertions are true of A_n:

(1) $f_{m,n}$ is of rank $r(n)$ where $r(0) = 1$ and $r(l + 1) = 2r(l) + 1$;

(2) A_n is locally finite;

(3) For any $m \in \omega$ and for any $\eta_0, \eta_1, \cdots, \eta_{r(n)-1} \in \omega_n$, we have

$$f_{m,n}(\eta_0, \cdots, \eta_{r(n)-1}) \leqq \cap (\{\eta_i \mid i \leqq r(n) - 1\} - \{\omega_s \mid s < n\})$$

and $f_{0,n}(\eta_0, \eta_0, \cdots, \eta_0) = \eta_0$;

(4) Given $\{\xi_k \mid k \in \omega\}$ a sequence of distinct members of ω_n, there exist an $m \in \omega$ and $k_0, k_1, \cdots, k_{r(n)} \in \omega$ so that $k_0 < \bigcap_{i=1}^{r(n)} k_i$ and

$$f_{m,n}(\eta_{k_1}, \cdots, \eta_{k_{r(n)}}) = \xi_{k_0}$$

where either $\eta_{k_i} = \xi_{k_i}$ or else $\eta_{k_i} \in \{\omega_s \mid s < n\}$.

It is clear that A_0 satisfies these conditions with $n = 0$.

Condition (3) will be used to obtain local finiteness, and condition (4) will assure that we have the descending chain condition for subalgebras. To see this suppose

$$A_n \supset_s B_0 \supset_s B_1 \supset_s \cdots .$$

Take $\xi_i \in B_i - B_{i+1}$. Then applying (4) to $\{\xi_i \mid i \in \omega\}$ we find that there is a $k_0 \in \omega$ for which $\xi_{k_0} \in B_{k_0+1}$, a contradiction.

We now proceed to construct A_{n+1} which satisfies conditions (1)—(4) with n replaced by $n + 1$. For each ξ with $\omega_n \leqq \xi < \omega_{n+1}$ we let ϕ_ξ map ξ onto ω_n in a one-to-one manner with ϕ_{ω_n} just the identity map on ω_n. For each $m \in \omega$ we define $f_{m,n+1}$ as follows: If $\omega_n \leqq \bigcap_{i=0}^{r(n)-1} \xi_i$; if $\eta_i < \xi_i$ for $i = 0, 1, \cdots, r(n) - 1$; if $\omega_n \leqq \gamma$; and if

$$\begin{aligned} &\phi_\gamma^{-1}(f_{m,n}(\phi_{\xi_0}(\eta_0), \cdots, \phi_{\xi_{r(n)-1}}(\eta_{r(n)-1}))) \\ &\quad \leqq \bigcap (\{\eta_0, \cdots, \eta_{r(n)-1}, \xi_0, \cdots, \xi_{r(n)-1}\} \\ &\qquad - \{\omega_s \mid s \leqq n\}); \end{aligned}$$

we define

$$\begin{aligned} &f_{m,n+1}(\xi_0, \cdots, \xi_{r(n)-1}, \eta_0, \cdots, \eta_{r(n)-1}, \gamma) \\ &\quad = \phi_\gamma^{-1}(f_{m,n}(\phi_{\xi_0}(\eta_0), \cdots, \phi_{\xi_{r(n)-1}}(\eta_{r(n)-1}))) . \end{aligned}$$

Otherwise we define

$$\begin{aligned} &f_{m,n+1}(\xi_0, \cdots, \xi_{r(n)-1}, \eta_0, \cdots, \eta_{r(n)-1}, \gamma) \\ &\quad = \bigcap \{\eta_0, \cdots, \eta_{r(n)-1}, \xi_0, \cdots, \xi_{r(n)-1}, \gamma\} . \end{aligned}$$

We let $A_{n+1} = \langle \omega_{n+1}; f_{m,n+1}, \omega_l \rangle_{\substack{m \in \omega \\ l \leqq n}}$.

It is clear that A_{n+1} satisfies conditions (1) and (3) of the induction hypothesis.

We now show that A_{n+1} is locally finite. Suppose B is a finite subset of ω_{n+1}. Let

$$\begin{aligned} &B_0 = B \cup \{\omega_s \mid s \leqq n\} , \\ &\vdots \\ &B_{k+1} = \{f_{m,n+1}(\xi_0, \cdots, \xi_{r(n+1)-1}) \mid m \in \omega \text{ and } \xi_0, \cdots, \xi_{r(n+1)-1} \in \textstyle\bigcup_{i \leqq k} B_i\} . \end{aligned}$$

Then $[B] = \bigcup_{k \in \omega} B_k$. In showing that $[B]$ is finite, we first show that each B_k is finite. This is true for $k = 0$. Assume that it is true for $k \leqq l$. Then $\bigcup_{i \leqq l} B_i$ is finite. Fix $\xi_0, \cdots, \xi_{r(n+1)-1} \in \bigcup_{i \leqq l} B_i$. Now we have

$$\begin{aligned} &\{f_{m,n+1}(\xi_0, \cdots, \xi_{r(n+1)-1}) \mid m \in \omega\} \\ &\quad \subseteqq \phi_{\xi_{r(n+1)-1}}^{-1}\{f_{m,n}(\phi_{\xi_0}(\xi_{r(n)}), \cdots, \phi_{\xi_{r(n)-1}}(\xi_{r(n+1)-1})) \mid m \in \omega\} \\ &\qquad \cup \{\xi_0 \cap \cdots \cap \xi_{r(n+1-1)}\} . \end{aligned}$$

However, this set is finite since A_n is locally finite. Hence B_{l+1} is finite, and by induction each B_k is finite. Now let $C_0 = B_0$ and

$$C_{k+1} = B_{k+1} - B_k .$$

Then $[B] = \bigcup_{k \in \omega} C_k$, and each C_{k+1} is finite. If $1 \leqq k < k'$ and if $C_k, C_{k'} \neq \emptyset$, then using (3) and the fact that $\{\omega_s \mid s \leqq n\} \subseteqq B_0$, we see that $\max C_{k'} < \max C_k$. Thus there are only finitely many $C_k \neq \emptyset$. Hence $[B]$ is finite.

Finally we show that A_{n+1} satisfies condition (4). Suppose we have $\{\xi_k \mid k \in \omega\}$ a sequence of distinct elements of ω_{n+1}. We consider two cases.

Case 1. There are infinitely many k's for which $\xi_k \in \omega_n$: Without loss of generality we assume that $\{\xi_k \mid k \in \omega\} \subseteqq \omega_n$. We then invoke the induction hypothesis to get an $m \in \omega$ and $k_0, k_1, \cdots, k_{r(n)} \in \omega$ so that $k_0 < \bigcap_{i=1}^{r(n)} k_i$ and $f_{m,n}(\eta_{k_1}, \cdots, \eta_{k_{r(n)}}) = \xi_{k_0}$ where either $\eta_{k_i} = \xi_{k_i}$ or else $\eta_{k_i} \in \{\omega_s \mid s < n\}$. But then we have

$$\begin{aligned} f_{m,n+1}(\omega_n, \cdots, \omega_n, \eta_{k_1}, \cdots, \eta_{k_{r(n)}}, \omega_n) &= \phi_{\omega_n}^{-1}(f_{m,n}(\phi_{\omega_n}(\eta_{k_1}), \cdots, \phi_{\omega_n}(\eta_{k_{r(n)}}))) \\ &= f_{m,n}(\eta_{k_1}, \cdots, \eta_{k_{r(n)}}) \\ &= \xi_{k_0} . \end{aligned}$$

This completes the proof in this case.

Case 2. At most finitely many of the ξ_k's are less than ω_n: Without loss of generality we assume that $\{\xi_k \mid k \in \omega\} \subseteqq \omega_{n+1} - \omega_n$. We pick $k_0 < k_1 < \cdots$ so that $\xi_{k_0} < \xi_{k_1} < \cdots$. For each $i \in \omega$, we let $\pi_i = \phi_{\xi_{k_{i+1}}}(\xi_{k_i})$. Now consider $\{\pi_i \mid i \in \omega\}$. If for some $i, j \in \omega$ we have $i < j$ and $\pi_i = \pi_j$, then

$$\begin{aligned} f_{0,n+1}(\xi_{k_{j+1}}, \cdots, \xi_{k_{j+1}}, \xi_{k_j}, \cdots, \xi_{k_j}, \xi_{k_{i+1}}) &= \phi_{\xi_{k_{i+1}}}^{-1}(f_{0,n}(\phi_{\xi_{k_{j+1}}}(\xi_{k_j}), \cdots, \phi_{\xi_{k_{j+1}}}(\xi_{k_j}))) \\ &= \phi_{\xi_{k_{i+1}}}^{-1}(f_{0,n}(\pi_j, \cdots, \pi_j)) \\ &= \phi_{\xi_{k_{i+1}}}^{-1}(\pi_j) \\ &= \phi_{\xi_{k_{i+1}}}^{-1}(\pi_i) \\ &= \phi_{\xi_{k_{i+1}}}^{-1}\phi_{\xi_{k_{i+1}}}(\xi_{k_i}) \\ &= \xi_{k_i} , \end{aligned}$$

and we're through. Thus we may assume that $\{\pi_i \mid i \in \omega\}$ is a sequence of distinct elements of ω_n. Applying the induction hypothesis again, we get an $m \in \omega$ and $i_0, i_1, \cdots, i_{r(n)} \in \omega$ so that $i_0 < \bigcap_{j=1}^{r(n)} i_j$ and

$$f_{m,n}(\eta_{i_1}, \cdots, \eta_{i_{r(n)}}) = \pi_{i_0}$$

where either $\eta_{i_j} = \pi_{i_j}$ or else $\eta_{i_j} \in \{\omega_s \mid s < n\}$.
Now let

$$\beta_{i_j} = \begin{cases} \xi_{k_{i_j}} & \text{if } \eta_{i_j} = \pi_{i_j} , \\ \eta_{i_j} & \text{otherwise} , \end{cases}$$

and let

$$\sigma_{i_j} = \begin{cases} \xi_{k_{i_j+1}} & \text{if } \beta_{i_j} = \xi_{k_{i_j}} , \\ \omega_n & \text{otherwise} . \end{cases}$$

Then $\phi_{\sigma_{i_j}}(\beta_{i_j}) = \eta_{i_j}$ in any case. This gives

$$\begin{aligned} f_{m,n+1}&(\sigma_{i_1}, \cdots, \sigma_{i_{r(n)}}, \beta_{i_1}, \cdots, \beta_{i_{r(n)}}, \xi_{k_{i_0+1}}) \\ &= \phi^{-1}_{\xi_{k_{i_0+1}}}(f_{m,n}(\phi_{\sigma_{i_1}}(\beta_{i_1}), \cdots, \phi_{\sigma_{i_{r(n)}}}(\beta_{i_{r(n)}}))) \\ &= \phi^{-1}_{\xi_{k_{i_0+1}}}(f_{m,n}(\eta_{i_1}, \cdots, \eta_{i_{r(n)}})) \\ &= \phi^{-1}_{\xi_{k_{i_0+1}}}(\pi_{i_0}) \\ &= \phi^{-1}_{\xi_{k_{i_0+1}}}\phi_{\xi_{k_{i_0+1}}}(\xi_{k_{i_0}}) \\ &= \xi_{k_{i_0}} . \end{aligned}$$

Since each $\sigma_{i_j}, \beta_{i_j}$ is a ξ_{k_i} with $i > i_0$ or is in $\{\omega_s \mid s \leqq n\}$, this is the desired result. This completes the proof of Theorem 3.3.

COROLLARY 3.4. *For $n \in \omega$ there is a locally finite algebra of power ω_n which has one commutative binary operation and satisfies the descending chain condition for subalgebras.*

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Received June 16, 1968. This work was supported in part by a National Science Foundation Grant GP-7252. These results are taken from the author's doctoral thesis, supervised by Professor B. Jónsson at Vanderbilt University.

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