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VECTOR VALUED ORLICZ SPACES GENERALIZED N-FUNCTIONS, I.

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The theory of Orlicz spaces generated by N-functions of a real variable is well known. On the other hand, as was pointed out by Wang, this same theory generated by N-functions of more than one real variable has not been discussed in the literature. The purpose of this paper is to develop and study such a class of generalized N-functions (called GN-functions) which are a natural generalization of the functions studied by Wang and the variable N-functions by Portnov. In second part of this study we will utilize GN-functions to define vector-valued Orlicz spaces and examine the resulting theory.

This paper is divided into five sections. In § 2, we define and examine some basic properties of GN-functions. A generalized delta condition is introduced and characterized in § 3. In § 4 and § 5 we present, respectively, the theory of an integral mean for GN-functions and the concept of a conjugate GN-function. A complete bibliography on Orlicz spaces, N-functions, and related material can be found in [4, 8]. The study of variable N-functions by Portnov can be found in [6, 7] and the study of nondecreasing N-functions by Wang in [9].

2. *GN*-functions. In what follows T will denote a space of points with σ-finite measure and E^n n dimensional Euclidean space.

DEFINITION 2.1. Let $M(t, x)$ be a real valued nonnegative function defined on $T \times E^n$ such that

(i) $M(t, x) = 0$ if and only if $x = 0$ for all $t \in T, x \in E^n$,

(ii) $M(t, x)$ is a continuous convex function of x for each t and a measurable function of t for each x,

(iii) For each $t \in T$, $\lim_{|x|=\infty} \dfrac{M(t, x)}{|x|} = \infty$, and

(iv) There is a constant $d \geqq 0$ such that

$$(*) \qquad \inf_t \inf_{c \geqq d} k(t, c) > 0$$

where

$$k(t, c) = \frac{\underline{M}(t, c)}{\bar{M}(t, c)},\ \bar{M}(t, c) = \sup_{|x|=c} M(t, x)\ ,$$
$$\underline{M}(t, c) = \inf_{|x|=c} M(t, x)$$

and if $d > 0$, then $\bar{M}(t, d)$ is an integrable function of t. We call a function satisfying properties (i)—(iv) a *generalized N-function* or a *GN-function.*

GN-functions are coordinate independent and are not necessarily symmetric. Therefore, such functions as $M(t, x) = x_1^2 + x_2^2 + (x_1 - x_2)^2$ which are not nondecreasing (as defined in [9]) are allowed in the class of GN-functions. The next theorem illustrates this point.

THEOREM 2.1. *If $M(t, x)$ is a GN-function and A is an orthogonal linear transformation defined on E^n with range in E^n, then $\tilde{M}(t, x) = M(t, Ax)$ is a GN-function.*

Properties (i)—(iv) when applied to $\tilde{M}(t, x)$ follow immediately from the same properties for $M(t, x)$ (see [8, Th. 8.1]).

The next theorem characterizes a part of property (iv) in Definition 2.1 and provides a means of comparing function values at different points for GN-functions when $|x|$ is large.

THEOREM 2.2. *A necessary and sufficient condition that* (*) *hold is that if $|x| \leqq |y|$, then there exist constants $K \geqq 1$ and $d \geqq 0$ such that $M(t, x) \leqq KM(t, y)$ for each $t \in T$ and $|x| \geqq d$.*

If (*) is true, then there exists a constant $d \geqq 0$ such that $l(t) = \inf_{c \geqq d} k(t, c) > 0$ for each t in T. By definition of $k(t, c)$ this means

$$M(t, y) \geqq \underline{M}(t, |y|) \geqq l(t)\bar{M}(t, |y|) \tag{2.2.1}$$

for any y such that $|y| = c \geqq d$. On the other hand, if $d \leqq |x| \leqq |y|$, then the convexity of $M(t, x)$ and $M(t, 0) = 0$ yields

$$\bar{M}(t, |y|) \geqq \sup_{|z|=|x|} M(t, z) . \tag{2.2.2}$$

Combining (2.2.1) and (2.2.2) we arrive at

$$M(t, y) \geqq l(t) \sup_{|z|=|x|} M(t, z) \geqq K^{-1}M(t, x)$$

whenever $d \leqq |x| \leqq |y|$ where $K^{-1} = \inf_t l(t) > 0$

The converse follows easily from the condition in the theorem.

It is interesting to note that if $M(t, x)$ is a GN-function, then $2\hat{M}(t, x) = M(t, x) + \tilde{M}(t, x)$ is also a GN-function where $\tilde{M}(t, x)$ is defined as in Theorem 2.1. This means we can construct a symmetric (in x) GN-function from one which does not possess this property. For, if $\tilde{M}(t, x) = M(t, -x)$, then $\hat{M}(t, x)$ is clearly symmetric in x.

Property (iv) of Definition 2.1 provides the condition which allows a natural generalization from N-functions of a real variable to those of several real variables. Let us observe that the function $\bar{M}(t, c)$ is also a GN-function of a real nonnegative variable c. On the other hand, $M(t, c)$ need not even be convex in c.

Since $\underline{M}(t, c) \leqq M(t, x) \leqq \bar{M}(t, c)$ for each x such that $|x| = c$, we would like to find a GN-function which bounds $\underline{M}(t, c)$ from below for all c. If $d = 0$ in Theorem 2.2, then $K^{-1}\bar{M}(t, c)$ would do.

One might accomplish the construction of such a function by taking the supremum of a class of convex functions bounding $\underline{M}(t, c)$ from below. This function would be convex. However, this class may be empty. The next theorem shows that this is not the case whenever $M(t, x)$ is a GN-function. The construction employed can be applied to more general settings than exist here.

THEOREM 2.3. *If $M(t, x)$ is a GN-function and $\underline{M}(t, c)$ is defined as above, then there exists a GN-function $R(t, c)$ such that $R(t, c) \leqq \underline{M}(t, c)$ for all $c \geqq 0$.*

Since $\underline{M}(t, c)$ satisfies property (iii) of Definition 2.1, given any $d > 0$ there is a $c_0 > 0$ such that $\underline{M}(t, c) \geqq dc$ whenever $c \geqq c_0$. Let us define the function

$$P(t, c) = \begin{cases} \sup\limits_{\substack{0 < w \leqq 1 \\ cw \geqq c_0}} \dfrac{\underline{M}(t, cw)}{w} & \text{if } c \geqq c_0 \\ \underline{M}(t, c) & \text{if } 0 \leqq c < c_0 \,. \end{cases}$$

Then it is easy to show that (i) $P(t, ac) \leqq aP(t, c)$ for $0 \leqq a \leqq 1$, (ii) $\{P(t, c)/c\}$ is a nondecreasing function of c, and (iii) $P(t, c)$ is finite for each c. We now obtain the desired function $R(t, c)$ by defining

$$R(t, c) = \int_0^c Q(t, s)ds$$

where

$$Q(t, c) = \begin{cases} \dfrac{P(t, c)}{c} & \text{if } c \geqq c_0 \\ \dfrac{cP(t, c_0)}{c_0^2} & \text{if } 0 \leqq c < c_0 \,. \end{cases}$$

We have immediately that

$$R(t, c) \leqq cQ(t, c) = P(t, c) \leqq \underline{M}(t, c) \,.$$

If is not difficult to show that $R(t, c)$ is also a GN-function.

3. Delta condition. In this section a generalized growth condition is defined for GN-functions. This growth or delta condition generalizes that definition usually given for a real variable N-function (e.g., see [4, 6, 7]).

DEFINITION 3.1. We say a GN-function $M(t, x)$ satisfies a *Δ-condition* if there exist a constant $K \geqq 2$ and a non-negative measurable function $\delta(t)$ such that the function $\bar{M}(t, 2\delta(t))$ is integrable over the domain T and such that for almost all t in T we have

$$(**) \qquad M(t, 2x) \leqq KM(t, x)$$

for all x satisfying $|x| \geqq \delta(t)$.

We say a GN-function satisfies a *Δ_0-condition* if it satisfies a Δ-condition with $\delta(t) = 0$ for almost all t in T.

In Definition 3.1 we could have used any constant $l > 1$ in place of the scalar 2 in (**). This is the basis of the next theorem which gives an equivalent definition to that employed in 3.1.

THEOREM 3.1. *A GN-function $M(t, x)$ satisfies a Δ-condition if and only if given any $l > 1$ there exists a constant $K_l \geqq 2$ and a nonnegative measurable function $\delta(t)$ such that $\bar{M}(t, 2\delta(t))$ is integrable over T and such that for almost all t in T we have*

$$(3.1.1) \qquad M(t, lx) \leqq K_l M(t, x)$$

whenever $|x| \geqq \delta(t)$.

Suppose $M(t, x)$ satisfies a Δ-condition and $l > 1$. We choose m so large that $2^m \geqq l$. Then by convexity and our assumption of a Δ-condition there is a $K \geqq 2$ and measurable $\delta(t) \geqq 0$ such that for almost all t in T

$$M(t, lx) \leqq M(t, 2^m x) \leqq K^m M(t, x)$$

whenever $|x| \geqq \delta(t)$. Therefore (3.1.1) holds with $K_l = K^m$. The converse follows as easily.

Whenever we deal with convex functions of several variables the concept of a one sided directional derivative plays an important role. The next result utilizes such a function, so we define it now.

DEFINITION 3.2. For each t in T the *directional derivative of a GN-function $M(t, x)$ in a direction y* is defined by

$$M'(t, x; y) = \lim_{h=0^+} \frac{M(t, x + hy) - M(t, x)}{h} .$$

The important properties of directional derivatives of convex functions of several variables which will be needed can be found in [3, 8]. Using the directional derivative defined above, the next result characterizes the delta condition and generalizes similar results given in [4, 6, 7].

THEOREM 3.2. *A GN-function $M(t, x)$ satisfies a Δ-condition if and only if there exists a nonnegative measurable function $\delta(t)$ such that $\bar{M}(t, 2\delta(t))$ is integrable over T and a constant $c > 1$ such that for almost all t in T*

$$\frac{M'(t, x; x)}{M(t, x)} < c \tag{3.2.1}$$

whenever $|x| \geqq \delta(t)$. Moreover, if (3.2.1) *holds, then for almost all t in T and for each x such that $|x| \geqq \delta(t)$ we have*

$$M(t, px) < M(t, x)p^c \tag{3.2.2}$$

for all $p > 1$.

Suppose $M(t, x)$ satisfies a Δ-condition. Then, by convexity (see, [8, Th. 5.3]), we must have for some $K \geqq 2$ and $\delta(t) \geqq 0$

$$KM(t, x) \geqq M(t, 2x) \geqq M(t, x) + M'(t, x; x)$$

whenever $|x| \geqq \delta(t)$. This means (3.2.1) holds with $c = K$.

Conversely, suppose (3.2.1) holds. We choose s such that $s \geqq 1$. Then, by assumption, there is a constant $c > 1$ and $\delta(t) > 0$ such that for almost all t in T

$$\frac{M'(t, sx; sx)}{M(t, sx)} > c \tag{3.2.3}$$

whenever $|x| \geqq \delta(t)$. On the other hand, we have

$$\begin{aligned} \frac{d}{ds} M(t, sx) &= \lim_{h=0^+} \frac{M(t, sx + hx) - M(t, sx)}{h} \\ &= M'(t, sx; x) . \end{aligned} \tag{3.2.4}$$

Since $M'(t, sx; sx) = sM'(t, sx; x)$ for all $s \geqq 0$, we obtain from (3.2.3) using (3.2.4) that

$$\log M(t, sx) \Big|_{s=1}^{s=2} = \int_1^2 \frac{M'(t, sx; x)}{M(t, sx)} ds < c \int_1^2 \frac{ds}{s} = \log 2^c . \tag{3.2.5}$$

Therefore, upon simplifying the last inequality, we arrive at

$$M(t, 2x) < 2^c M(t, x)$$

whenever $|x| \geqq \delta(t)$ proving the first part of the theorem.

The last inequality (3.2.2) in the theorem is obtained from (3.2.5) whenever we integrate over $1 \leqq s \leqq p$, $p > 1$.

Inequality (3.2.2) states that GN-functions which satisfy a Δ-condition do not grow faster than a power function along any ray passing through the origin. Let us also observe that any function $M(t, x)$ defined on $T \times E^n$ which is either subadditive or a positive homogeneous (of degree one) convex function always satisfies a Δ_0-condition.

4. Generalized mean functions. In this section an integral mean will be defined for GN-functions. We will show under what conditions the mean function is a GN-function and satisfies a Δ-condition. Moreover, we examine how the minimizing points in the definition of the mean function affect a basic property of the ordinary integral mean.

Let us begin with a definition.

DEFINITION 4.1. For each t in T and $h > 0$ let

$$M_h(t, x) = \int_{E^n} M(t, x + y) J_h(y) dy$$

where $J_h(y)$ is a nonnegative, c^∞ function with compact support in a ball of radius h such that $\int_{E^n} J_h(y)dt = 1$. Moreover, let x_0 be any point (depending on h, t) which satisfies the inequality

$$M_h(t, x_0) \leqq M_h(t, x)$$

for all x in E^n. Then the function $\hat{M}_h(t, x)$ defined for each t in T and $h > 0$ by

$$\hat{M}_h(t, x) = M_h(t, x + x_0) - M_h(t, x_0)$$

is called a *mean function for $M(t, x)$ relative to the minimizing point x_0.*

The next theorem shows under what condition $\hat{M}_h(t, x)$ is a GN-function.

THEOREM 4.1. *If $M(t, x)$ is a GN-function for which $\bar{M}(t, c)$ is integrable in t for each c, then $\hat{M}_h(t, x)$ is a GN-function.*

We will show this result by justifying conditions (i)—(iv) of Definition 2.1. By hypothesis and the choice of x_0, we have for each h, $\hat{M}_h(t, x) \geqq 0$ and $\hat{M}_h(t, 0) = 0$. On the other hand, if $x \neq 0$, then

$M(t, x) > 0$, and hence there is constant h_0 such that

$$a = \inf_{|z| \leqq h_0} M(t, x + z) > 0 .$$

However, since $M(t, x) = 0$ if and only if $x = 0$, the minimizing points x_0 tend to zero as h tends to zero. Therefore, we can choose $g_0 \leqq h_0$ such that if $h \leqq g_0$, then $M(t, x_0+y) < a$ for all y for which $|x_0+y| < h$. For this g_0 we obtain the inequality

$$M(t, x + x_0 + y) \geqq \inf_{|z| \leqq g_0} M(t, x + z) \geqq a > M(t, x_0 + y)$$

whenever $|x_0 + y| \leqq g_0$. This means for some $h \leqq g_0$ we have

$$M_h(t, x + x_0) > M_h(t, x_0)$$

or $\hat{M}_h(t, x) > 0$ if $x \neq 0$ which proves property (i).

Properties (ii) and (iii) for $\hat{M}_h(t, x)$ follow easily from the same properties for $M(t, x)$. Let us now show (iv). By assumption, there is a constant $d \geqq 0$ such that

$$l(t)\bar{M}(t, c) \leqq \underline{M}(t, c) \tag{4.1.1}$$

for all $c \geqq d$. Furthermore, it is not difficult to show that for all c we have

$$\bar{M}(t, c) \geqq \sup_{|x| \leqq c} M(t, x) \tag{4.1.2}$$

and for some fixed z,

$$\inf_{|x| \geqq c} M(t, x + z) \leqq \inf_{|x| = c} M(t, x + z) . \tag{4.1.3}$$

Using (4.1.2), we obtain for each t in T that

$$\begin{aligned} l(t) \sup_{|x|=c} M(t, z) &\leqq l(t) \sup_{|w| < c + |x_0+y_1|} M(t, w) \\ &\leqq l(t) \sup_{|w| = c + |x_0+y_1|} M(t, w) \end{aligned} \tag{4.1.4}$$

where $z = x + x_0 + y$. On the other hand, by (4.1.1) and (4.1.3), we achieve

$$\begin{aligned} l(t) \sup_{|w| = c + |x_0+y_1|} M(t, w) &\leqq \inf_{|w| = c + |x_0+y_1|} M(t, w) \\ &< \inf_{|x| \geqq c} M(t, x + x_0 + y) \\ &< \inf_{|x| = c} M(t, x + x_0 + y) . \end{aligned} \tag{4.1.5}$$

If we combine (4.1.4) and (4.1.5), then for all $c \geqq d$ we arrive at

$$l(t) \sup_{|x|=c} M(t, x + x_0 + y) \leqq \inf_{|x|=c} M(t, x + x_0 + y) .$$

From this inequality we obtain

$$\begin{aligned}(4.1.6)\qquad \inf_{|x|=c} \hat{M}_h(t, x) &\geqq \int_{E^n} \inf_{|x|=c} \{M(t, x + x_0 + y) - M(t, x_0 + y)\}J_h(y)dy \\ &\geqq \int_{E^n} \{l(t) \sup_{|x|=c} M(t, x + x_0 + y) - M(t, x_0 + y)\}J_h(y)dy\end{aligned}$$

and

$$(4.1.7)\qquad \sup_{|x|=c} \hat{M}_h(t, x) \leqq \int_{E^n} \sup_{|x|=c} M(t, x + x_0 + y)J_h(y)dy .$$

Moreover, since $\lim_{c=\infty} \sup_{|x|=c} M(t, x + x_0 + y) = \infty$ for fixed x_0, y such that $|y| \leqq h$, given $K_1(t) = 2 \sup_{|y|\leqq h} M(t, x_0 + y)/\inf_t l(t)$ there is a $d_1 > 0$ such that if $c \geqq d_1$, then $\sup_{|x|=c} M(t, x + x_0 + y) \geqq K_1$. Therefore, using (4.1.6) and (4.1.7), we achieve the inequalities

$$\begin{aligned}(4.1.8)\qquad \frac{\inf_{|x|=c} \hat{M}_h(t, x)}{\sup_{|x|=c} \hat{M}_h(t, x)} &\geqq l(t) - \frac{\sup_{|y|\leqq h} M(t, x_0 + y)}{\inf_{|y|\leqq h} \sup_{|x|=c} M(t, x + x_0 + y)} \\ &\geqq l(t) - \frac{1}{2} \inf_t l(t)\end{aligned}$$

for all $c \geqq d_0 = \max(d, d_1, |x_0|)$. Taking the infimum of both sides of (4.1.8) over t, shows the first part of property (iv). To show the latter part, assume $d_0 > 0$. Then $\sup_{|x|=d_0} \hat{M}_h(t, x)$ is integrable over t in T since it is bounded by the integrable function $\bar{M}(t, d_2)$ where $d_2 = d_0 + |x_0| + h$. This proves property (iv) and the theorem.

In the next theorem we show under what condition $\hat{M}_h(t, x)$ satisfies a Δ-condition.

THEOREM 4.2. *If $M(t, x)$ is a GN-function satisfying a Δ-condition and for which $\bar{M}(t, c)$ is integrable in t for each c, then $\hat{M}_h(t, x)$ satisfies a Δ-condition.*

It suffices to show that $M_h(t, x)$ satisfies a Δ-condition. For, $\hat{M}_h(t, x)$ is the sum of a constant and a translation of $M_h(t, x)$ and neither of these operations affects the growth condition. Let us observe first that if $|x| \geqq 2, |y| \leqq h \leqq 1$, then $|2x + y| \leqq 3|x + y|$. Hence, by Theorem 2.2, there are constants $K \geqq 1$ and $d_1 \geqq 0$ such that

$$M_h(t, 2x) \leqq K \int_{E^n} M(t, 3(x + y))J_h(y)dy$$

for all x such that $|x| \geqq d_2 = \max(d_1, 2)$. On the other hand, by Theorem 3.1, there is a constant $K_3 \geqq 2$ and $\delta(t) \geqq 0$ such that for almost all t in T

$$\int_{E^n} M(t, 3(x + y))J_h(y)dy \leqq K_3 M_h(t, x)$$

for all x, y such that $|x + y| \geqq \delta(t)$ where $|y| \leqq h$. Combining the above two inequalities we achieve

$$M_h(t, 2x) \leqq KK_3 M_h(t, x)$$

for all $|x| > \max(d_2, \delta(t) + h) = \delta_1(t)$. Since $\bar{M}(t, 2\delta_1(t))$ is integrable over T, this yields the integrability of $\bar{M}_h(t, 2\delta_1(t))$ proving the theorem.

For each t in T and each x in E^n it is known that $\lim_{h=0} M_h(t, x) = M(t, x)$. However, the same property does not hold in general for $\hat{M}_h(t, x)$. This is the point of the next theorem.

THEOREM 4.3. *For each $h > 0$ let x_0^h be the minimizing point of $M_h(t, x)$ defining $\hat{M}_h(t, x)$. Then for each t in T and each x in E^n, there exists $K(t, x)$ such that*

$$\lim_{h=0} \hat{M}_h(t, x) = M(t, x) + K(t, x) \lim_{h=0} |x_0^h| .$$

By definition of $\hat{M}_h(t, x)$ we can write

$$\begin{aligned} &|\hat{M}_h(t, x) - M(t, x)| \\ &\leqq \int_{E^n} |M(t, x + x_0^h + y) - M(t, x_0^h + y) - M(t, x)| J_h(y)dy . \end{aligned} \tag{4.3.1}$$

However, we know that

$$\begin{aligned} &|M(t, x + x_0^h + y) - M(t, x_0^h + y) - M(t, x)| \\ &\quad \leqq |M(t, x + x_0^h + y) - M(t, x)| \\ &\qquad + |M(t, x_0^h + y) - M(t, y)| + |M(t, y)| . \end{aligned} \tag{4.3.2}$$

Moreover, since $M(t, x)$ is a convex function, it satisfies a Lipshitz condition on compact subsets of E^n (see, [8, Th. 5.1]). Therefore, there exist $K_1(t, x)$ and $K_2(t, x)$ such that

$$|M(t, x + x_0^h + y) - M(t, x)| \leqq K_1(t, x) |x_0^h + y| \tag{4.3.3}$$

and

$$|M(t, x_0^h + y) - M(t, y)| \leqq K_2(t, x) |x_0^h| . \tag{4.3.4}$$

If we combine (4.3.3) and (4.3.4) with (4.3.2) and if we substitute the resulting expression into (4.3.1), we achieve the inequality

$$\begin{aligned} |\hat{M}_h(t, x) - M(t, x)| &\leqq |x_0^h| (K_1(t, x) + K_2(t, x)) \\ &\quad + \int_{E^n} K_1(t, x) |y| J_h(y)dy + \int_{E^n} |M(t, y)| J_h(y)dy . \end{aligned}$$

Since the last two integrals on the right side tend to zero as h tends to zero, we prove the theorem by setting $K(t, x) = K_1(t, x) + K_2(t, x)$.

COROLLARY 4.3.1. *Suppose $M(t, x)$ is a GN-function such that $M(t, x) = M(t, -x)$. Then for each t in T and x in E^n,*

$$\lim_{h=0} M_h(t, x) = \hat{M}(t, x) .$$

This result is clear since $\lim_{h=0} |x_0^h| = 0$ if $M(t, x) = M(t, -x)$. In fact, if $M(t, x)$ is even in x then the $x_0^h = 0$ for all h.

For each t in T let A_h denote the set of minimizing points of $M_h(t, x)$ and let B represent the null space of $M(t, x)$ relative to points in E^n, i.e.,

$$B = \{y \text{ in } E^n: M(t, y) = 0\} .$$

If $M(t, x)$ is a GN-function, then $B = \{0\}$. For the sake of argument, let us suppose that $M(t, x)$ has all the properties of a GN-function except that $M(t, x) = 0$ need not imply $x = 0$. We will show the relationships that exist between A_h and B. This is the content of the next few theorems.

THEOREM 4.4. *The sets B and A_h are closed convex sets.*

This result follows from the convexity and continuity of $M(t, x)$ in x for each t in T.

THEOREM 4.5. *Let $B_e = \{x: M(t, x) < e\}$ for each t in T. Then given any $e > 0$, there is a constant $h_0 > 0$ such that $A_h \subseteqq B_e$ for each $h \leqq h_0$.*

Since $B \subseteqq B_e$, we can choose h_0 sufficiently small so that if x is in B, then $x + y$ is in B_e for all y such that $|y| \leqq h_0$. Let z be an arbitrary but fixed point in A_h, $h \leqq h_0$. Then

$$M_h(t, z) \leqq M_h(t, x)$$

for all x. Therefore, if x is in B, we have by our choice of h_0 that $M_h(t, z) < e$. Letting h tend to zero yields $M(t, z) < e$, i.e., z in B_e.

We have commented above that $A_h = \{0\}$ if $M(t, x) = M(t, -x)$. It is also true if $M(t, x)$ is strictly convex in x for each t in T.

THEOREM 4.6. *Suppose $M(t, x)$ is a GN-function which is strictly convex in x for each t. Then for each h, $A_h = \{0\}$.*

Suppose there exists $y_0 \neq x_0$ such that x_0, y_0 are in A_h. Let $z =$

$(x_0 + y_0)/2$. Then, since $M(t, x)$ is strictly convex, $M_h(t, x)$ is strictly convex in x. Therefore, we have

$$M_h(t, z) < \frac{1}{2} M_h(t, x_0) + \frac{1}{2} M_h(t, y_0) . \tag{4.6.1}$$

However, x_0, y_0 being in A_h reduces (4.6.1) to the inequality

$$M_h(t, z) < M_h(t, x)$$

for all x. This means z is in A_h and x_0, y_0 are not in A_h which is a contradiction. Hence, $x_0 = y_0$. Since $M(t, x)$ is a GN-function, $B = \{0\}$. In this case $x_0 = y_0 = 0$.

5. **Conjugate GN-functions.** In the study of Orlicz spaces the concept of a conjugate N-fuction plays a significant role. In particular, the definition of these linear spaces may involve a conjugate function. The study of convex functions of several variables and their related conjugate functions can be found in [1, 2, 3, 5].

In this section the concept of a generalized conjugate function is defined and some of its important properties are examined. Many of the standard results which hold for N-functions and conjugate functions of a real variable will be generalized here.

We begin with the main definition.

DEFINITION 5.1. Let $M(t, x)$ be a GN-function. Then we call $M^*(t, x)$ the *conjugate function* of $M(t, x)$ if for each t in T

$$M^*(t, x) = \sup_{z \text{ in } E^n} \{zx - M(t, z)\} . \tag{$+$}$$

The notation zx represents the scalar product of the vectors x and z.

Let us observe that if $zx \leqq 0$ in $(+)$, then $zx - M(t, z) \leqq 0$. This means we could, equivalently, restrict the definition to those z for which $zx \geqq 0$. Moreover, the equation $(+)$ yields immediately for each t in T that

$$zx \leqq M(t, z) + M^*(t, x) \tag{$++$}$$

for all z, x in E^n. Inequality $(++)$ could have been used as a definition of the conjugate function.

Fenchel [3] states that to every z in E^n such that $M'(t, z; y) < \infty$ for all y for which it is defined, there is at least one point x in E^n such that equality holds in $(++)$. However, by [8, Th. 5.2] when applied to GN-functions, we know for z in E^n that $M'(t, z; y) < \infty$ for all y. Therefore, the supremum in $(+)$ is attained for at least one point.

The next theorem gives a necessary and sufficient condition in order that equality hold in $(++)$.

THEOREM 5.1. *Let $M(t, x)$ be a GN-function for which $M'(t, x; y)$ is linear in y. Then, given any x_0, $z^i = M'(t, x_0; e_i)$ for all $i = 1, \cdots, n$ if and only if $zx_0 = M(t, x_0) + M^*(t, z)$ where $\{e_i\}$ is a basis for E^n.*

Clearly, if

$$zx_0 = M(t, x_0) + M^*(t, z)$$

for each t in T, then $z^i = M'(t, x_0; e_i)$ for each i. On the other hand, suppose $z^i = M'(t, x_0; e_i)$ for each $i = 1, \cdots, n$. Then, by convexity of $M(t, x)$ and linearity of $M'(t, x; y)$, we have for t in T

$$M(t, x) \geqq M(t, x_0) + z(x - x_0) \tag{5.1.1}$$

for all x in E^n. Rewriting (5.1.1) we obtain for all x in E^n

$$x_0 z - M(t, x_0) \geqq xz - M(t, x) .$$

Therefore, we have

$$x_0 z - M(t, x_0) \geqq \sup_x \{xz - M(t, x)\} = M^*(t, z)$$

or

$$x_0 z \geqq M(t, x_0) + M^*(t, z) . \tag{5.1.2}$$

Since $(++)$ always holds, combining (5.1.2) with $(++)$ shows that equality holds in (5.1.2).

The properties of GN-functions possessed by $M^*(t, x)$ are give in the next result.

THEOREM 5.2. *Let $M(t, x)$ be a GN-functions for which*

$$\lim_{|x|=0} \frac{M(t, x)}{|x|} = 0$$

for each t in T. Then $M^*(t, x)$ satisfies properties (i)—(iii) *of Definition* 2.1. *Moreover, if $M(t, x) = M(t, -x)$, then*

$$M^*(t, x) = M^*(t, -x) .$$

Condition (i) for $M^*(t, x)$ follows directly from the same condition for $M(t, x)$ and the equation in the hypothesis. Convexity follows from the inequality

$$M^*(t, ax + by) = \sup \{axz - aM(t, z) + byz + bM(t, z)\}$$
$$\leqq aM^*(t, x) + bM^*(t, y)$$

where $a + b = 1, a \geqq 0, b \geqq 0$. Measurability in t also follows from the same property for $M(t, x)$. Finally, if we substitute $z = kx/|x|$, $k>1$ into $(++)$ we arrive at

$$\frac{M^*(t, x)}{|x|} \geqq k - \frac{M\left(t, \frac{kx}{|x|}\right)}{|x|} . \tag{5.2.1}$$

However, $M(t, kx/|x|)$ is bounded on every compact set in E^n (see [8, Th. 2.5]). Letting $|x|$ tend to infinity in (5.2.1) results in property (iii).

Suppose $M(t, x)$ is an even function of x. Then

$$M^*(t, x) = \sup_z \{-zx - M(t, -z)\}$$
$$= \sup_z \{z(-x) - M(t, z)\} = M^*(t, -x) .$$

Finally, we give conditions when $M(t, x)$ is the conjugate function of $M^*(t, x)$.

THEOREM 5.3. *Suppose $M(t, x)$ is a GN-function for which $M'(t, x; y)$ is linear in y. Then $M(t, x)$ is the conjugate function of $M^*(t, x)$.*

Since $M(t, x)$ is convex in x and $M'(t, x; y)$ is linear in y, we achieve for any x, x_0 in E^n.

$$M(t, x) - M(t, x_0) \geqq M'(t, x_0; x - x_0)$$
$$\geqq M'(t, x_0; x) - M'(t, x_0: x_0)$$

from which it follows that

$$M'(t, x_0; x_0) - M(t, x_0) \geqq \sup_x \{xy - M(t, x)\} \tag{5.3.1}$$

where $y^i = M'(t, x_0; e_i)$ for each $i = 1, \cdots, n$ and $\{e_i\}$ basis vectors for E^n. On the other hand, it is clear that

$$M'(t, x_0; x_0) - M(t, x_0) \leqq \sup_x \{xy - M(t, x)\} \tag{5.3.2}$$

since $M'(t, x_0; x_0) = x_0 y$. Combining (5.3.1) and (5.3.2) we obtain the equation

$$x_0 y - M(t, x_0) = M^*(t, y) . \tag{5.3.3}$$

However, by $(++)$, we know that

$$x_0 z \leqq M(t, x_0) + M^*(t, z) \tag{5.3.4}$$

for all x_0, z in E^n. Rewriting (5.3.4) yields

$$M(t, x_0) \geqq \sup_z \{x_0 z - M^*(t, z)\} . \tag{5.3.5}$$

Since (5.3.3) holds for some y, it follows that

$$M(t, x_0) = x_0 y - M^*(t, y) \leqq \sup_z \{x_0 z - M^*(t, z)\} . \tag{5.3.6}$$

Therefore, combining (5.3.5) and (5.3.6) produces the desired result that

$$M(t, x_0) = \sup_z \{x_0 z - M^*(t, z)\} .$$

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