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ON INTERPOLATION OF q-VARIATE STATIONARY STOCHASTIC PROCESSES

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Let X_t be a q-variate stationary stochastic process. Let K be any set of t-values and let K' be the complement of K. If $s \in K'$ the problem of approximating X_s by linear combinations of the X_t's with $t \in K$ and limit of such linear combinations is considered. The best linear predictor and the mean square error matrix are evaluated in the following cases: (1) t takes on all real values, K consists of the integers (2) t is interger-valued, K consists of the odd integers.

Let $(X_k)_{-\infty}^{\infty}$, k an integer, be a q-variate weakly stationary stochastic process (SP). Let K be any subset of the set of integers and K' denote its complement in the set of all integers. Let $\mathscr{M}_K$ denote the (closed) subspace spanned by $X_k, k \in K$.

PREDICTION PROBLEM. Let $X_s, s \in K'$. Find $\hat{X}_s$ the projection of X_s onto $\mathscr{M}_K$ and the error matrix $(X_s - \hat{X}_s, X_s - \hat{X}_s)$[1].

In this paper we propose to solve the prediction problem for two cases:

(1) X_t, t real, is a q-variate stationary SP and K consists of the set of all integers.

(2) X_k, k an integer, is a q-variate stationary SP and K consists of the set of all odd integers.

For $q = 1$ these results have been previously obtained by A. M. Yaglom {cf. [12, p. 176]}.

In § 2 we will review the notion of absolute continuity of a matrix-valued signed measure with respect to another such measure {cf. [6]} and state a few results concerning the Hellinger-square integrability of matrix-valued measures. Our main result will be given in § 3.

2. Matrix-valued measures.

The problem of absolute continuity of a matrix-valued measure with respect to another matrix-valued measure was first posed by P. Masani in [4, p. 366]. Later J. B. Robertson and M. Rosenberg {cf. [6]} dealt with this question and were able to obtain a satisfactory solution to it. We will briefly review some of these results. Let Ω be any set and $\mathscr{B}$ be a σ-algebra of its subsets. M is said to be a $q \times r$ matrix-valued signed measure on $(\Omega, \mathscr{B})$ if for each $B \in \mathscr{B}$, $M(B)$ is a $q \times r$ matrix, with finite complex

[1] $(\ldots)$ denotes the inner product in the Hilbert space $\mathscr{H}^q$ containing the q-variate stochastic process X_k, k an integer.

entries, and $M(B) = \sum_{k=1}^{\infty} M(B_k)$, whenever $B_1, B_2, \cdots$ is a sequence of disjoint sets in $\mathscr{B}$ whose union is B. A $q \times q$ matrix-valued signed measure M is called a $q \times q$ matrix-valued measure if $M(B)$ is a nonnegative hermitian matrix for each $B \in \mathscr{B}$. Ψ is called a measurable $p \times q$ matrix-valued function on $(\Omega, \mathscr{B})$ if for each $\omega \in \Omega$, $\Psi(\omega)$ is a $p \times q$ matrix and if the entries of Ψ are measurable functions on $(\Omega, \mathscr{B})$. We say that a $q \times r$ matrix-valued signed measure is absolutely continuous (a.c.) with respect to (w.r.t.) a σ-finite nonnegative real-valued measure μ on $(\Omega, \mathscr{B})$ if the entries of M, M_{ij}'s are a.c. w.r.t. μ. We write $(dM/d\mu) = (dM_{ij}/d\mu)$ for the Radon-Nikodym derivative of M w.r.t. μ. The integral $N(B) = \int_B \Psi dM$ is defined by $N(B) = \int_B \Psi(dM/d\mu)d\mu$, where M is a.c. w.r.t. μ. It is easy to show that the definition of $N(B)$ is independent of the choice of μ.

DEFINITION 2.1. Let M and N be $p \times q$ and $r \times q$ matrix-valued signed measures on $(\Omega, \mathscr{B})$ respectively, μ be any σ-finite nonnegative real-valued measure on $(\Omega, \mathscr{B})$ such that M and N are a.c. w.r.t. μ. We say that N is a.c. w.r.t. M if

$$\kappa\left(\frac{dM}{d\mu}(\omega)\right) \subset \kappa\left(\frac{dN}{d\mu}(\omega)\right) \quad \text{a.e.} \quad \mu ,$$

where for each matrix A, $\kappa(A) = \{\alpha: A\alpha = 0\}$. It can be easily verified that this definition is independent of μ.

The following theorem is proved in [6].

THEOREM 2.2. *Let M and N be $p \times q$ and $r \times q$ matrix-valued signed measures on $(\Omega, \mathscr{B})$. Then*

(a) *N is* a.c. w.r.t. *M if and only if there exists a measurable $r \times p$ matrix-valued function Ψ on Ω such that for each $B \in \mathscr{B}$*

$$N(B) = \int_B \Psi dM$$

(b) *Let Φ and Ψ be measurable $r \times p$ matrix-valued functions on Ω. Then for each $B \in \mathscr{B}$, $\int_B \Phi dM = \int_B \Psi dM$ if and only if $\Phi J = \Psi J$* a.e. *μ, where J is the orthogonal projection matrix onto the range of $dM/d\mu$ and μ is any σ-finite nonnegative real-valued measure on $(\Omega, \mathscr{B})$* w.r.t. *which M is* a.c.

If N is a.c. w.r.t. M, then by Theorem 2.2 (a) there exists a measurable matrix-valued function Ψ such that for each $B \in \mathscr{B}$

$$N(B) = \int_B \Psi dM .$$

Ψ is called the Radon-Nikodym derivative of N. w.r.t. M and we will denote it by (dN/dM). We now review properties of Hellinger integrability of matrix-valued measures {cf. [9]}.

DEFINITION 2.3. Let M and N be $p \times q$ and $r \times q$ be matrix-valued measures on $(\Omega, \mathscr{B})$, F be a $q \times q$ matrix-valued measure on $(\Omega, \mathscr{B})$. We say that (M, N) is Hellinger-integrable w.r.t. F if $\int_\Omega (dM/d\mu)(dF/d\mu)^-(dN/d\mu)^* d\mu$[2] exists for some σ-finite nonnegative real-valued measure on $(\Omega, \mathscr{B})$, where $(dF/d\mu)^-$ denotes the generalized inverse of $(dF/d\mu)$ {cf. [5, p. 407]}. It is not hard to show that the existence and the value of this integral when it exists is independent of μ. We write

$$\int_\Omega \frac{dMdN^*}{dF} = \int_\Omega (dM/d\mu)(dF/d\mu)^-(dN/d\mu)^* d\mu \, .$$

The following theorem is needed later.

THEOREM 2.4. *Let* (i) *M and N be $p \times q$ and $r \times q$ matrix-valued signed measures on $(\Omega, \mathscr{B})$, F be a $q \times q$ matrix-valued measure on $(\Omega, \mathscr{B})$.*

(ii) *M or N, say M, be* a.c. w.r.t. F. *Then (M, N) is Hellinger integrable* w.r.t. *F if and only if the Lebesgue integral $\int_\Omega (dM/dF)dN^*$ exists. In case these integrals exist, their values are equal.*

Proof. Let μ be any σ-finite nonnegative real-valued measure on $(\Omega, \mathscr{B})$ w.r.t. which M, N and F are a.c. Since M is a.c. w.r.t. F then by Theorem 2.2 there exists a measurable $p \times q$ matrix-valued function Ψ on Ω such that for each $B \in \mathscr{B}$

$$M(B) = \int_B \Psi dF, \Psi J = \Psi \quad \text{a.e.} \quad \mu \, , \tag{1}$$

where J is the orthogonal projection matrix onto the range of $dF/d\mu$. If $\int_\Omega dMdN^*/dF$ exists, then from the following chain of equality it follows that $\int_\Omega (dM/dF)dN^*$ exists and the two integrals are equal

$$\begin{aligned} \int_\Omega \frac{dMdN^*}{dF} &= \int_\Omega (dM/d\mu)(dF/d\mu)^-(dN/d\mu)^* d\mu \\ &= \int_\Omega \Psi (dF/d\mu)(dF/d\mu)^-(dN/d\mu)^* d\mu \\ &= \int_\Omega \Psi (dN/d\mu)^* d\mu = \int_\Omega (dM/dF)dN^* \, , \end{aligned} \tag{2}$$

[2] denotes the adjoint operation.

where the first equality is a consequence of Definition 2.3, the second is a consequence of (1), the third one is a consequence of $(dF/d\mu)(dF/d\mu)^- = J$ and (1) and the last two are consequences of (1). Similarly if $\int_\Omega (dM/dF)dN^*$ exists from (2) it follows that $\int_\Omega dMdN^*/dF$ exists and these integrals are equal.

3. Interpolation of a stationary SP **with continuous time parameter.** Let X_t, t real, be a q-variate weakly stationary SP with the spectral distribution $q \times q$ matrix-valued function F defined on $(-\infty, \infty)$. Suppose that the process has been observed at the time points $k = \cdots, -1, 0, 1, \cdots$ and we wish to estimate X_t where t is not an integer. First we state a lemma whose proof is immediate.

LEMMA 3.1. *Let K be the set of all integers. Then*

(a) *for each $\lambda \in (0, 2\pi]$ the series*

$$\sum_{k \in K} [F(\lambda + 2k\pi) - F(2k\pi)]$$

converges and defines a $q \times q$ nonnegative hermitian matrix-valued function $G(\cdot)$ on $(0, 2\pi]$.

(b) *$G(\cdot)$ is monotone nondecreasing on $(0, 2\pi]$ and*

$$G(2\pi) \leqq \lim_{\lambda \to \infty} F(\lambda) .$$

(c) *For each $\lambda \in (0, \pi]$ and each fixed real t the series*

$$\sum_{k \in K} e^{-2ik\pi t}[F(\lambda + 2k\pi) - F(2k\pi)]$$

converges and defines a $q \times q$ matrix-valued function $G_t(\cdot)$ on $(0, 2\pi]$.

(d) *G_t is of bounded variation on $(0, 2\pi]$ and the variation of $G_t \leqq G(2\pi)$.*

(e) *G and G_t define $q \times q$ matrix-valued measure and signed measure on the Borel family of subsets of $(0, \pi]$ respectively.*

(f) *G_t is* a.c. w.r.t. *G.*[3]

We are now ready to state the main result of this notion. For standard terminology and notation of q-variate stationary processes used in Theorem 3.2 we refer to [4] and [8].

THEOREM 3.2. (i) *Let X_t, t real, be a q-variate weakly stationary SP with the spectral representation $X_t = \int_{-\infty}^{\infty} e^{-it\lambda}E(d\lambda)X_0$, the spectral*

[3] By "G_t is a.c. w.r.t. G" we mean that the $q \times q$ matrix-valued signed measure M_t generated by G_t is absolutely continuous w.r.t. the $q \times q$ matrix-valued measure M generated by G.

distribution function F defined on $(-\infty, \infty)$.

(ii) *Let K denote the set of all integers, $\mathscr{M}_K$ the (closed) subspace spanned by X_t, $t \in K$ and for each $t \notin K$ let $\hat{X}_t$ be the projection of X_t onto $\mathscr{M}_K$. Then*

(a) *There exists a $q \times q$ matrix-valued function $\Psi_t \in L_{2,F}$*[4] *such that $\hat{X}_t = \int_{-\infty}^{\infty} \Psi_t(\lambda)E(d\lambda)X_0$, the function Ψ_t is periodic of period 2π.*

(b) *If $G(\cdot)$ and $G_t(\cdot)$ are the matrix-valued functions defined in Lemma 3.1, then*

$$\Psi_t(\lambda) = e^{-it\lambda}(dG_t/dG)(\lambda) \quad \text{a.e.} \quad F .$$

(c) *The interpolation error matrix $\sum_t = (X_t - \hat{X}_t, X_t - \hat{X}_t)$ is given by*

$$\sum_t = \frac{1}{2\pi}\int_0^{2\pi}(I - dG_t/dG)dF(I - dG_t/dG)^* ,$$

where I is the identity matrix of order $q \times q$.

Proof. (a) Let V denote the isomorphism mapping from $L_{2,F}$ onto $\mathscr{M}$ the (closed) subspace spanned by the SP X_t {cf. [7, p. 297]}. Since $\mathscr{M}_K \subseteqq \mathscr{M}$, there exists a $\Psi_t \in L_{2,F}$ such that

$$\hat{X}_t = \int_{-\infty}^{\infty} \Psi_t E(d\lambda)X_0 . \tag{1}$$

From the definition of V it follows that for each $k \in K$

$$Ve^{-ik\lambda}I = X_k . \tag{2}$$

Since for each $k \in K$, $e^{-ik\lambda}$ has period 2π and since $\hat{X}_t \in \mathscr{M}_K$, from (1) and (2) it follows that $\Psi_t(\lambda)$ is periodic and has period 2π.

(b) By (a) we have

$$\hat{X}_t = \int_{-\infty}^{\infty} \Psi_t(\lambda)E(d\lambda)X_0 .$$

It then immediately follows that

$$\int_{-\infty}^{\infty}[e^{-it\lambda}I - \Psi_t(\lambda)]dF(\lambda)e^{-ik\lambda} = (X_t - \hat{X}_t, X_k) = 0 \tag{3}$$

for each $k \in K$.

Since $\Psi_t \in L_{2,F}$, $\Psi_t \in L_{2,G} \cap L_{2,G_t}$. Hence

$$\int_0^{2\pi} e^{-ik\lambda}[e^{-i\lambda t}dG_t(\lambda) - \Psi_t(\lambda)dG(\lambda)]$$
$$= \int_0^{2\pi} e^{-ik\lambda}e^{-i\lambda t}dG_t(\lambda) - \int_0^{2\pi} e^{-ik\lambda}\Psi_t(\lambda)dG(\lambda) .$$

[4] $L_{2,F}$ is an abbreviation for $L_2((-\infty, \infty), \mathscr{B}, F)$, {cf. [7, p. 295]}.

$$\text{The first term} = \int_0^{2\pi} e^{-ik\lambda}e^{-i\lambda t}d\Big(\sum_{n\in K} e^{-2in\pi t}[F(\lambda+2n\pi)-F(2n\pi)]\Big)$$

$$= \sum_{n\in K}\int_0^{2\pi} e^{-ik\lambda}e^{-it\lambda}d(e^{-2in\pi t}[F(\lambda+2n\pi)-F(2n\pi)])$$

$$= \sum_{n\in K}\int_{2n\pi}^{2(n+1)\pi} e^{-ik(\mu-2n\pi)}e^{-i(\mu-2n\pi)}e^{-2in\pi t}d[F(\mu)-F(2n\pi)]$$

$$= \sum_{n\in K}\int_{2n\pi}^{2(n+1)\pi} e^{-ik\mu}e^{it\mu}d[F(\mu)-F(2n\pi)]$$

$$= \int_{-\infty}^{\infty} e^{-it\lambda}e^{-ik\lambda}dF(\lambda)\ .$$

Also since $\Psi_t(\lambda)$ is periodic of period 2π,

$$\int_0^{2\pi} e^{-ik\lambda}\Psi_t(\lambda)dG(\lambda) = \int_0^{2\pi} e^{-ik\lambda}\Psi_t(\lambda)d\Big[\sum_{n\in K} F(\lambda+2n\pi)-F(2n\pi)\Big]$$

$$= \sum_{n\in K}\int_0^{2\pi} e^{-ik\lambda}\Psi_t(\lambda)d[F(\lambda+2n\pi)-F(2n\pi)]$$

$$= \sum_{n\in K}\int_{2n\pi}^{2(n+1)\pi} e^{-ik\lambda}\Psi_t(\lambda)d[F(\lambda)-F(2n\pi)]$$

$$= \int_{-\infty}^{\infty} e^{-ik\lambda}\Psi_t(\lambda)dF(\lambda)\ .$$

Hence

$$\int_0^{2\pi} e^{-ik\lambda}[e^{-i\lambda t}dG_t(\lambda) - \Psi_t(\lambda)dG(\lambda)] = \int_{-\infty}^{\infty} [e^{-i\lambda t}I - \Psi_t(\lambda)]e^{-ik\lambda}dF(\lambda)\ . \tag{4}$$

By (3) and (4) we get that

$$\int_0^{2\pi} e^{-ik\lambda}e^{-i\lambda t}dG_t(\lambda) = \int_0^{2\pi} e^{-ik\lambda}\Psi_t(\lambda)dG(\lambda)\ . \tag{5}$$

Since by (5) the Fourier coefficients of the matrix-valued signed measures $M(B) = \int_B e^{-it\lambda}dG_t(\lambda)$ and $N(B) = \int_B \Psi_t(\lambda)dG(\lambda)$, B is a Borel subset of $(0, 2\pi]$, are the same, it follows that for each Borel subset B of $(0, 2\pi]$

$$M(B) = \int_B e^{-it\lambda}dG_t(\lambda) = \int_B \Psi_t(\lambda)dG(\lambda)\ .$$

Now let μ be any σ-finite nonnegative real-valued measure on $(\Omega, \mathscr{B})$ w.r.t. G is a.c. Then automatically G_t is a.c. w.r.t. μ, because G_t is a.c. w.r.t. G. Therefore we have

$$M(B) = \int_B e^{-it\lambda}(dG_t/dG)(\lambda)dG(\lambda) = \int_B \Psi_t(\lambda)dG(\lambda)\ . \tag{6}$$

From (6) and Theorem 2.2 (b) it follows that

$$(7) \qquad e^{-it\lambda}(dG_t/dG)J = \Psi_t J \quad \text{a.e.} \quad \mu ,$$

where J is the orthogonal projection matrix onto the range of $dG/d\mu$. Since G is a.c. w.r.t. μ, F is also a.c. w.r.t. μ. Because $\Psi_t \in L_{2,F}$ a simple calculation shows that $\Psi_t J \in L_{2,F}$ and that

$$(8) \qquad \Psi_t J = \Psi_t \quad \text{a.e.} \quad \mu .$$

But $(dG_t/dG)J = \Psi_t J$, therefore $(dG_t/dG)J \in L_{2,F}$. This easily implies that $(dG_t/dG) \in L_{2,F}$ and

$$(9) \qquad (dG_t/dG)J = (dG_t/dG) \quad \text{a.e.} \quad \mu .$$

From (7), (8) and (9) we have

$$e^{-it\lambda}(dG_t/dG) = \Psi_t \quad \text{a.e.} \quad \mu$$

i.e.

$$e^{-it\lambda}(dG_t/dG) = \Psi_t \quad \text{a.e.} \quad F .$$

(c) We have $X_t = \int_{-\infty}^{\infty} e^{-it\lambda}E(d\lambda)X_0$ and

$$\hat{X}_t = \int_{-\infty}^{\infty} e^{-it\lambda}(dG_t/dG)(\lambda)E(d\lambda)X_0 .$$

Hence from the isometry theorem {cf. [7, p. 297]} we obtain

$$\Sigma_t = (X_t - \hat{X}_t, X - \hat{X}_t) = \frac{1}{2\pi}\int_0^{2\pi}(I - dG_t/dG)dF(I - dG_t/dG)^* .$$

As a special case of Theorem 3.2 we have the following result concerning a q-variate stationary stochastic process with discrete time parameter.

THEOREM 3.3. *Let*

(i) *X_k, k an integer, be a q-variate weakly stationary SP with the spectral representation $X_k = \int_0^{2\pi} e^{-ik\lambda}dE(\lambda)X_0$ with spectral distribution F defined on $(0, 2\pi]$.*

(ii) *Let K be the set of all odd integers, $\mathscr{M}_K$ the (closed) subspace spanned by X_k, $k \in K$ and let for each $k \in K$, $\hat{X}_k$ denote the projection of X_k onto $\mathscr{M}_K$. Then*

(a) *there exists a $q \times q$ matrix-valued function $\Psi_k \in L_{2,F}$ such that $\hat{X}_k = \int_0^{2\pi}\Psi_k(\lambda)E(d\lambda)X_0$. $e^{i\lambda}\Psi_k$ is periodic of period π.*

(b) *Ψ_k is given by*

$$\Psi_k(\lambda) = e^{-ik\lambda}\frac{d[F(\cdot) + e^{-i\pi}F(\cdot + \pi)]}{d[F(\cdot) + F(\cdot + \pi)]}(\lambda) \quad \text{a.e.} \quad F \ \textit{if} \ \lambda \in (0, \pi]$$

$$\Psi_k(\lambda) = e^{-i(k+1)\pi}\Psi_k(\lambda - \pi) \quad \text{a.e.} \quad F \ \textit{if}\ \lambda \in (\pi, 2\pi]\,.$$

(c) *The interpolation error matrix* $\Sigma_k = (X - \hat{X}_k, X - \hat{X}_k)$ *is given by*

$$\sum_k = \frac{2}{\pi}\int_0^\pi \frac{dF(\lambda+\pi)}{d[F(\lambda)+F(\lambda+\pi)]}\,dF(\lambda)$$
$$= \frac{2}{\pi}\int_0^\pi \frac{dF(\lambda+\pi)dF(\lambda)}{d[F(\lambda)+F(\lambda+\pi)]}\,,$$

where the first is a Lebesgue integral and the last one is a Hellinger integral.

Proof. Since the proof of (a) is similar to that of Theorem 3.2 (a), we proceed to sketch the proof of parts (b) and (c). Let for each real t

$$S(t) = \int_0^{4\pi} \exp\left\{-i\left(t - \frac{1}{2}\right)\lambda\right\} dF\left(\frac{\lambda}{2}\right)$$

and $Y(t)$ be a q-variate stationary stochastic process with correlation function $S(t)$. Note that for each integer n

$$S(n/2) = R(n-1)\,. \tag{1}$$

Using results (b) and (c) of Theorem 3.2 for the processes $Y(t)$, from (1), part (b) and the first equation for $\sum_k$ easily follow. The second equation for $\sum_k$ is obtained from Theorem 2.4, since $dF(\lambda+\pi)$ is a.c. w.r.t. $d[F(\lambda)+F(\lambda+\pi)]$.

REFERENCES

1. J. L. Doob, *Stochastic Processes*, Wiley, 1953.
2. V. Grenander, and M. Rosenblatt, *Statistical Analysis of Stationary Time Series*, Wiley, 1957.
3. A. N. Kolmogorov, *Stationary sequences in Hilbert space*, Bull. Math. Univ., Moscow **2**, (1941).
4. P. R. Masani, *Recent trends in multivariate prediction theory*, Multivariate Analysis, Academic Press, 1966, 351-382.
5. R. A. Penrose, *A generalized inverse for matrices*, Proc. Camb. Phil. Soc. **51** (1955), 406-413.
6. J. B. Robertson and M. Rosenberg, *The decomposition of matrix-valued measures* (to appear in Illinois J. Math.)
7. M. Rosenberg, *The square-integrability of matrix-valued functions with respect to a non-negative hermitian measure*, Duke Math. J. (1964), 291-298.
8. Yu. A. Rozanov, *Stationary random processes*, Moscow, (Russian), 1963; English transl. by A. Feinstein, Holden Day, 1967.
9. H. Salehi, *The Hellinger square-integrability of matrix-valued measures with respect to a non-negative hermitian measure*, Ark. for Mat. (1967), 299-303.

10. ———, *Application of the Hellinger integrals to q-variate stationary stochastic processes*, Ark. för Mat. (1967), 305-311.
11. N. Wiener and P. Masani, *The prediction theory of multivariate stochastic processes*, I, Acta Math. **98** (1957), 111-150.
12. A. M. Yaglom, *On linear interpolation of stationary stochastic sequences and processes*, Usp. Math. Navk. **4** (1949), 173-178; Amer. Math. Soc. Sel. Transl. in Math. Stat. **4** (1963).

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