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REGULAR AND IRREGULAR MEASURES ON GROUPS AND DYADIC SPACES

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It is generally known that if X is a σ-compact metric space, then every Borel measure on X is regular. It is not difficult to prove a slightly stronger result, namely that the same conclusion holds if X is a Hausdorff space in which every open subset is σ-compact (I.6 below). The converse is not generally true, even for compact Hausdorff spaces; a counter-example appears here under IV. 1. However, it will be shown in § II that every nondegenerate Borel measure on a nondiscrete locally compact group is regular if and only if the group is σ-compact and metrizable. A similar theorem, proved in § III, holds for dyadic spaces: every Borel measure on such a space is regular if and only if the space is metric.

The result for groups depends on two structure theorems which are proved here: every nonmetrizable compact connected group contains a nonmetrizable connected Abelian subgroup (II.10), and every nonmetrizable locally compact group contains a nonmetrizable compact totally disconnected subgroup (II.11).

In § III, it seems that the separable case requires special attention: a theorem is proved which has as a corollary that every separable dyadic space is a continuous image of $\{0, 1\}^c$ (III.3 and III.4), and one lemma (III.6) uses a weakened version of the continuum hypothesis.

I. Regular and irregular measures.

1. Let X be a topological space, $\mathbf{M}$ a σ-algebra of subsets of X, and μ a (countably additive, nonnegative) measure function whose domain is $\mathbf{M}$. The system $(X, \mathbf{M}, \mu)$ is called *regular measure space* and μ is called a *regular measure* in case

(1) $\mu C < \infty$ for all compact $C \in \mathbf{M}$;

(2) $\mu S = \inf\{\mu U: U \text{ open}, U \in \mathbf{M}, U \supset S\}$ for all $S \in \mathbf{M}$;

(3) $\mu U = \sup\{\mu C: C \text{ compact}, C \in \mathbf{M}, C \subset U\}$ for all open $U \in \mathbf{M}$.

For lack of a better term, a measure μ will be called *totally regular* if it satisfies the more exclusive definition of regularity favored by some authors (e.g., Halmos in [5]), namely:

$$\mu S = \sup\{\mu C: C \text{ compact}, C \in \mathbf{M}, C \subset S\}$$
$$= \inf\{\mu U: U \text{ open}, U \in \mathbf{M}, U \supset S\} \text{ for all } S \in \mathbf{M}.$$

REMARK 2. According to [7], (10.30) and (10.31), any σ-finite regular measure on a Hausdorff space is totally regular; the proof as given is for Radon measures but almost exactly the same argument will work for any regular measure.

3. A measure μ will be called *irregular* if
(1) μ is not regular;
(2) $\mu C < \infty$ for all compact $C \in \mathbf{M}$;
(3) μ is nondegenerate: i.e., μ has values other than 0 and ∞.

4. Let X be a topological space. $\mathbf{B}(X)$ is defined to be the smallest σ-algebra containing the closed subsets of X. A *Borel measure* on X is a measure defined on $\mathbf{B}(X)$ which assigns finite measure to each compact member of $\mathbf{B}(X)$.

5. *Note*. Research on nonregular measures has appeared in [12], [13], and [14], and examples of irregular Borel measures are to be found in [5] and [7]; see II.2 and IV.2 below.

It is clear that the construction of a nonregular degenerate measure on a space which is not σ-compact presents no problem: simply assign measure 0 to sets which are contained in σ-compact sets, and measure ∞ to other sets.

LEMMA 6. *Let X be a topological space such that every open subset of X is the union of countably many closed sets. Let μ be a σ-finite Borel measure on X. Then*

$$\mu B = \sup\{\mu F : F \text{ closed}, \ F \subset B\} = \inf\{\mu U : U \text{ open}, \ U \supset B\}$$

for all $B \in \mathbf{B}(X)$.

(This result is due to E. Zakon [16].)

THEOREM 7. *Let X be a Hausdorff space. If every open subset of X is σ-compact, then every Borel measure on X is totally regular.*

Proof. This follows easily from the preceding lemma.

COROLLARY 8. *Every Borel measure on a σ-compact metric space is regular.*

II. Locally compact groups. All topological groups in this section are assumed to be Hausdorff.

THEOREM 1. *Let G be a locally compact group which is neither σ-compact nor discrete. Then G admits an irregular Borel measure.*

Proof. Let λ be a left Haar measure on G. For $B \in \mathbf{B}(G)$, define

$\nu B = \sup\{\lambda C: C \text{ compact}, C \subset S\}$. To show that ν is a nondegenerate Borel measure is a routine exercise. Now let H be an open σ-compact subgroup of G, and let A be a subset of G containing exactly one element of each left coset of H. Clearly A is closed and, by the argument in [6], (16.14), $\lambda A = \infty$ but A is locally λ-null, i.e., $\nu A = 0$. Since $\nu U = \infty$ for each neighborhood U of A, ν is irregular.

[See IV.2 for an example.]

2. Let Ω denote the first uncountable ordinal and Γ denote an arbitrary ordinal with no countable cofinal subsets, following the standard convention whereby an ordinal is identified with the set of its predecessors.

THEOREM. *Let $X_0 = \Gamma$ with the order topology. For $B \in \mathbf{B}(X_0)$, define*

$$\mu B = \begin{cases} 1 \text{ if } B \text{ contains a closed cofinal subset of } X_0 \\ 0 \text{ otherwise}. \end{cases}$$

Then μ is an irregular Borel measure on X_0.

Proof. The argument is essentially the same as that required for the special case $\Gamma = \Omega$, which appears as an exercise in [5] (p. 231). Using a variation of the "interlacing lemma" as in [1], it can be shown that the intersection of countably many closed cofinal sets is cofinal; thus a member of $\mathbf{B}(X_0)$ has measure 1 if and only if its complement has measure 0 and the union of countably many sets of measure 0 has measure 0 also, so that μ is indeed a Borel measure. The measure is irregular as $\mu X_0 = 1$ while $\mu C = 0$ for every compact subset C of X_0.

COROLLARY 3. *Let $X_1 = \Gamma \cup \{\Gamma\}$ with the order topology and let X be a T_1 space. Suppose that there is a continuous function h from X_1 into X such that $h^{-1}\{(h\Gamma)\}$ is not cofinal in X_0. Then X admits a finite irregular Borel measure.*

Proof. It is easy to verify that $h^{-1}(B) \cap X_0$ is in $\mathbf{B}(X_0)$ whenever B is in $\mathbf{B}(X)$. Let μ be the irregular measure defined in II.2 and define ν on $\mathbf{B}(X)$ by $\nu B = \mu(h^{-1}(B) \cap X_0)$; evidently ν is a Borel measure, which is irregular since $\nu\{h(\Gamma)\} = 0$ but $\nu U = 1$ for each neighborhood U of $h(\Gamma)$.

EMBEDDING THEOREM 4. *Let $X_1 = \Omega \cup \{\Omega\}$ with the order topology. X_1 is homeomorphic to a subspace of $\{0, 1\}^\Omega$ (with the product topology).*

Proof. For $\alpha \in X_1$, define $h(\alpha)$ in $\{0, 1\}^{\Omega}$ by

$$[h(\alpha)]_\beta = h_\beta(\alpha) = \begin{cases} 0 \text{ if } \alpha \leqq \beta \\ 1 \text{ if } \alpha > \beta . \end{cases}$$

Evidently, h is one-to-one. Each coordinate function h_β is continuous from X_1 into $\{0, 1\}$; thus h is continuous.

COROLLARY 5. *Any space which contains* $\{0, 1\}^{\Omega}$ *as a closed subspace admits a finite irregular Borel measure.*

REMARK 6. According to a theorem of Ivanovskii *et. al.* ([6], (9.15)), every nonmetrizable compact totally disconnected group is homeomorphic to $\{0, 1\}^m$ for some uncountable m. By II.5, every such group therefore admits an irregular Borel measure; this is a special case of corollary II.12 below. In order to prove II.12 in general, we show that every nonmetrizable locally compact group has a nonmetrizable compact totally disconnected subgroup.

LEMMA 7. *Let G be a locally compact group with identity e and closed normal subgroup H. If H and G/H are both metrizable, then so is G.*

Proof. This follows from (8.5) of [6], together with the continuity of the natural homomorphism.

LEMMA 8. *Let G be a torsion-free Abelian group of rank r. Then there exists a subgroup K of G such that G/K is a torsion group and* $\text{card}\,(G/K) \geqq r + 1$. *If G is uncountable, then* $\text{card}\,(G/K) = \text{card}\,(G)$.

Proof. Let L be a maximal independent subset of G, let K_0 be the subgroup generated by L, and (using additive notation) let $K = 2K_0$. By the maximality of L, G/K_0 and therefore G/K are torsion. If α and β are distinct elements of L, then $\alpha \notin K$ and $\alpha - \beta \notin K$ by the independence of L. Thus $\text{card}\,(G/K) \geqq \text{card}\,(L) + 1 = r + 1$. A standard argument (e.g., see [4], p. 32) shows that if G is uncountable then $\text{card}\,(G) = r$, so that $\text{card}\,(G/K) = \text{card}\,(G)$.

THEOREM 9. *Let G be a nonmetrizable compact connected Abelian group. Then G contains a nonmetrizable compact totally disconnected subgroup.*

Proof. Let Γ be the dual group of G; Γ is an uncountable discrete torsion-free Abelian group and thus, by the previous lemma, has a subgroup K such that Γ/K is an uncountable torsion group. Let $H = \{g \in G: \gamma(g) = 1 \text{ for all } \gamma \in K\}$; H is a subgroup of G, topologically isomorphic to the dual group of Γ/K, and therefore compact, nonmetrizable, and totally disconnected. (See [6], (23.25), (24.26), and (24.15).)

LEMMA 10. *Let G be a nonmetrizable compact connected group. Then G contains a nonmetrizable compact connected Abelian group.*

Proof. Let H be any maximal Abelian subgroup of G; according to [9], H is connected and every maximal Abelian subgroup of G is a conjugate of H. Let V be any intersection of countably many neighborhoods of e. By [6], (8.7), V contains a compact normal subgroup N of G such that G/N is metrizable. Suppose $N \cap H = \{e\}$; then $N \cap H' = \{e\}$, where H' is any other maximal Abelian subgroup of G. Consequently $N = \{e\}$, which is impossible since G is not metrizable. Thus $V \cap H \supset N \cap H \neq \{e\}$, and thus H is not metrizable.

THEOREM 11. *Let G be a nonmetrizable locally compact group. Then G contains a nonmetrizable compact totally disconnected subgroup.*

Proof. (1) Assume G is compact. Let C be the component of e in G. If C is metrizable, then there exists a compact normal subgroup H of G such that $H \cap C = \{e\}$ and G/H is metrizable; by II.7, H is not metrizable. The natural homomorphism $g \to g^C$ is a topological isomorphism of H onto CH/C, a subgroup of the totally disconnected group G/C ([6], (7.3)); H is therefore totally disconnected. If C is not metrizable, then C contains a nonmetrizable compact totally disconnected subgroup, by II.9 and II.10.

(2) Now suppose G is not compact. By part (1), we have only to show that G has a nonmetrizable compact subgroup. Let H be an open compactly generated subgroup; by [6], (8.5) and (8.7), H is not metrizable and has a compact normal subgroup N such that H/N is metrizable, and thus N is not metrizable.

COROLLARY 12. *Every nonmetrizable locally compact group admits a finite irregular Borel measure, concentrated on a compact totally disconnected subgroup.*

This follows from the remark in II.6. Combining II.12 with II.1 I.8, we have:

THEOREM 13. *Let G be a nondiscrete locally compact group.*

Then every nondegenerate Borel measure on G is regular if and only if G is σ-compact and metrizable.

III. Dyadic spaces.

1. A *dyadic space* is a Hausdorff space which is the image, under a continuous mapping, of $\{0, 1\}^A$ for some set A, where $\{0, 1\}$ is discrete and the product has the product topology. According to a standard theorem, every compact metric space is a dyadic space; thus a dyadic space is any Hausdorff space which is a continuous image of a product of compact metric spaces. Recent interesting papers on dyadic spaces include [2] and [3], which contain references to earlier writings.

THEOREM 2. *Let X be a dyadic space. Then every Borel measure on X is regular if and only if X is metric.*

Proof. If X is metric, then every Borel measure on X is regular, by I.8; to prove the converse statement, some preliminary results have to be established, as follows:

THEOREM 3. *Let X be a dyadic space and D a dense subset of X. Then there is a continuous function from $\{0, 1\}^{2^D}$ onto X.*
[See [3], Theorem 1, for related result.]

Proof. Let f be a continuous function from $\{0, 1\}^A$ onto X. Choose $E \subset \{0, 1\}^A$ such that $f \mid E$ is one-to-one and $f(E) = D$. Define an equivalence relation $\sim$ on A as follows: $\alpha \sim \beta$ in case $x_\alpha = x_\beta$ for all $x \in E$. Define $u(\alpha) = \{x \in E : x_\alpha = 1\}$ and $U = \{u(\alpha) : \alpha \in A\}$; clearly $u(\alpha) = u(\beta)$ if and only if $\alpha \sim \beta$. Define a mapping g from $\{0, 1\}^U$ into $\{0, 1\}^A$ by $[g(t)]_\alpha = g_\alpha(t) = t_{u(\alpha)}$ for each $t = (t_{u(\alpha)})$ in $\{0, 1\}^U$ and each α in A. Each g_α is continuous from $\{0, 1\}^U$ into $\{0, 1)$, thus g is continuous, and $f \circ g$ is a continuous mapping from $\{0, 1\}^U$ into X. The image of g in $\{0, 1\}^A$ is the set $\{x : x_\alpha = x_\beta \text{ whenever } \alpha \sim \beta\}$, which contains E. Thus $f \circ g$ is a continuous function from $\{0, 1\}^U$ onto a dense compact subset of X, which must be X itself. Now card $U \leqq 2^{\operatorname{card} E} = 2^{\operatorname{card} D}$; thus there is a continuous function from $\{0, 1\}^{2^D}$ onto X.

COROLLARY 4. *A dyadic space is separable if and only if it is a continuous image of $\{0, 1\}^c$.*

Proof. By [11], Theorem 1, $\{0, 1\}^c$ (and every continuous image thereof) is separable. The converse follows from III.3.

THEOREM 5. *Let X be a topological space and let $\{X_\alpha: \alpha < \Gamma\}$ be a nondecreasing transfinite sequence of proper closed subsets of X with $\bigcup X_\alpha$ dense in X. Let A be a subset of Γ. Then the following statements are equivalent*:

(1) *A is cofinal in Γ.*

(2) $\bigcup\{X_\alpha: \alpha \in A\} = \bigcup\{X_\alpha: \alpha < \Gamma\}$.

(3) *$\bigcup\{X_\alpha: \alpha \in A\}$ is dense in X.*

Proof. It is clear that each of (1) and (2) implies the statement following it. To show that (3) implies (1), suppose A has an upper bound $\alpha_0 < \Gamma$. Then $\bigcup\{X_\alpha: \alpha \in A\} \subset X_{\alpha_0}$, which is a proper closed subset of X, and so $\bigcup\{X_\alpha: \alpha \in A\}$ is not dense in X, contradicting (3).

[*Note*: To prove the next lemma, we assume a weakened version of the continuum hypothesis, namely that $c = \aleph_j$ for some $j = 1, 2, \cdots$].

LEMMA 6. *Let X be a nonmetric dyadic space. Let Γ be the smallest ordinal such that X contains a nonmetric subspace which is a continuous image of $\{0, 1\}^\Gamma$. Then Γ does not have a countable cofinal subset.*

Proof. Let X_Γ be the continuous image of $\{0, 1\}^\Gamma$ referred to in the hypothesis; Γ is uncountable since X_Γ is not metric. If X_Γ is separable, then card$(\Gamma) \leqq c$, so by the note above, Γ does not have a countable cofinal subset. On the other hand, if X_Γ is not separable, let f be a continuous function from $\{0, 1\}^\Gamma$, onto X_Γ and for $\alpha < \Gamma$ let

$$F_\alpha = \{y: y \in \{0, 1\}^\Gamma,\ y_\beta = 0 \text{ for all } \alpha \leqq \beta < \Gamma\} .$$

Let $X_\alpha = f(F_\alpha)$. $\bigcup F_\alpha$ is dense in $\{0, 1\}^\Gamma$ and thus $\bigcup X_\alpha$ is dense in X_Γ. Now, for each $\alpha < \Gamma$, F_α is homeomorphic to $\{0, 1\}^\alpha$, thus X_α is a compact metric (hence closed and separable) subspace of X_Γ. Since X_Γ is not separable, it is impossible for the union of countably many X_α to be dense in X_Γ, and therefore by III.5, Γ does not have a countable cofinal subset.

Proof of III.2 (Conclusion). Suppose X is a nonmetric dyadic space. Let Γ and X_Γ be as in III.6; let f, F_α, and $X_\alpha(\alpha < \Gamma)$ be as defined above. Since Γ has no countable cofinal subset, $\bigcup X_\alpha \neq X_\Gamma$ (by [2], Corollary 1.). Choose $h(\Gamma)$ in $\{0, 1\}^\Gamma$ such that $f(h(\Gamma)) \notin \bigcup X_\alpha$. Let $A = \{\alpha < \Gamma: h(\Gamma)_\alpha = 1\}$; then we have

$$\begin{aligned} h(\Gamma) \in \{0, 1\}^\Gamma - \bigcup F_\alpha &= \bigcap\nolimits_\alpha (\{0, 1\}^\Gamma - F_\alpha) \\ &= \bigcap\nolimits_\alpha \{y: y_\beta = 1 \text{ for some } \alpha \leqq \beta < \Gamma\} , \end{aligned}$$

thus A is cofinal in Γ and so has no countable cofinal subset. Let $X_1 = A \cup \{\Gamma\}$ with the well-ordering inherited from $\Gamma \cup \{\Gamma\}$ and with the order topology. Define a function h from X_1 into $\{0, 1\}^\Gamma$ coordinatewise by

$$[h(\alpha)]_\beta = h_\beta(\alpha) = \begin{cases} 1 \text{ if } \beta \in A \text{ and } \beta < \alpha \\ 0 \text{ otherwise} \end{cases}$$

for $\alpha \in A$ and $\beta < \Gamma$; $h(\Gamma)$ has already been defined. By the definition of the topologies of X_1 and $\{0, 1\}$, each coordinate function h_β is continuous, thus, h is continuous. It is obvious that h is one-to-one. Now $f \circ h$ is a continuous function from X_1 into X, and $(f \circ h)^{-1}(\Gamma) = \{\Gamma\}$, for if $\alpha \in A$ then $f \circ h(\alpha) \in X_\alpha$, but $f \circ h(\Gamma) = f(h(\Gamma)) \notin X_\alpha$. Since A has no countable cofinal subset, II.3 applies and X admits an irregular Borel measure.

COROLLARY 7. *Every nonmetrizable locally compact group admits a finite irregular Borel measure, concentrated on a compact subgroup.*

Proof. II.11(2) of this paper shows that a nonmetrizable locally compact group has a nonmetrizable compact subgroup. According to a theorem of Kuzminov, [6] p. 106, every compact group is a dyadic space.

The reader will note that this corollary is a less precise version of II.12.

8. (A concluding remark on finite irregular measures.) A measure ν is *continuous* if each point $x \in X$ is an element of a set of measure; 0; ν is *atomic* if it has an *atom*, i.e., a measurable set A, such that $\nu A > 0$ and such that, when S is a measurable subset of A, either $\nu S = 0$ or $\nu S = \nu A$.

THEOREM. *Let X be a Hausdorff space and let $(X, \mathbf{M}, \nu)$ be a measure space with $\nu = \nu_1 + \nu_2$, where ν_1 is a finite continuous atomic measure. Then ν is irregular.*

Proof. We assume without loss of generality that $\nu_2 = 0$ and $\{x\} \in \mathbf{M}$ for all $x \in X$. Let A be an atom and let $\mathbf{C} = \{C: C \in \mathbf{M}, C$ compact, $C \subset A, \nu C = \nu A\}$. If $\mathbf{C} = \emptyset$, ν is irregular, according to I.2. If $\mathbf{C} \neq \emptyset$, then $\bigcap \mathbf{C} \neq \emptyset$ as $\mathbf{C}$ is a collection of closed compact sets with the finite intersection property. Let $x \in \bigcap \mathbf{C}$. If C is a compact measurable subset of $A - \{x\}$, then $\nu C = 0$ as $C \subset A$ and $x \notin C$. But $\nu(A - \{x\}) = \nu A > 0$, thus ν is irregular.

It will be noted that the finite irregular Borel measures described in § II and § III are atomic. The author is not aware of any finite irregular measures that do not have the properties described in the theorem above.

IV. Examples.

1. Let X be the one-point compactification of a discrete space of cardinality $\aleph_1$. Evidently, every subset of X is either open or closed (or both), and thus a member of $\mathbf{B}(X)$. Every Borel measure μ on X is therefore a finite measure defined on all subsets of X, and so, by a theorem of Ulam [15],

$$\begin{aligned}\mu B &= \sum_x \mu(\{x\}) \qquad (x \in B)\\ &= \sup\{\mu A : A \text{ is finite and } A \subset B\}\\ &= \inf\{\mu S : X - S \text{ is finite and } B \subset S\} .\end{aligned}$$

Thus μ is totally regular. However, any uncountable subset of $X - \{\infty\}$ is an open set which is not σ-compact.

This example provides a comment on II.3; we cannot weaken the hypothesis by eliminating the condition that $h^{-1}\{(h\Gamma)\}$ not be cofinal, even if we substitute the condition that X have non-σ-compact open subsets. Let $\Gamma = \Omega$ and take X to be the one-point compactification of the isolated ordinals in Ω. Define $h: X_1 \to X$ by

$$h(\alpha) = \begin{cases}\infty & \text{if } \alpha \text{ is a limit ordinal}\\ \alpha & \text{otherwise} .\end{cases}$$

Then h is continuous but X, as just noted, admits no irregular Borel measure.

2. Let R denote the reals with the usual topology and R_d the reals with the discrete topology; let $G = R_d \times R$ with the product topology. For $r \in R_d$ and $S \subset G$, set $S(r) = \{x : (r, x) \in S\}$. Note that if $B \in \mathbf{B}(G)$, then $B(r) \in \mathbf{B}(R)$ for all $r \in R_d$. Define

$$\nu B = \sum_r \lambda(B(r)) \qquad (r \in R_d)$$

where λ denotes Lebesgue measure on R. Clearly ν is a nondegenerate Borel measure, which is irregular since there is a set $A = R_d \times \{0\}$ such that $\nu A = 0$ but $\nu U = \infty$ for any neighborhood U of A.

This example, which has appeared in [7] (Exercise 12.58), provides a specific illustration for Theorem II.1. For let

$$\mu S = \inf\{\nu U : U \text{ open}, \ U \supset S\} ;$$

it can be shown (see [10], 2.22) that μ is a Haar measure for G, and that

$$\nu B = \sup\{\mu C : C \text{ compact}, \ C \subset B\} .$$

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