PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 1, 1969

ON UNICITY OF CAPACITY FUNCTIONS

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Sario's capacity function of a closed subset γ of the ideal boundary is known to be unique if γ is of positive capacity. The present paper will determine the number of capacity functions of γ in terms of the Heins harmonic dimension when γ has zero capacity, under the assumption that γ is isolated. This includes the special case where γ is the ideal boundary.

1. Capacity functions. Denote by β the ideal boundary of an open Riemann surface R in the sense of Kerékjártó-Stoïlow. We consider a fixed nonempty closed subset $\gamma \subset \beta$ which is *isolated* from $\delta = \beta - \gamma$. Throughout this paper D will denote a fixed parametric disk about a fixed point $\zeta \in R$ with a fixed local parameter z and the uniqueness is always referred to this fixed triple (ζ, D, z). Here we do not exclude the case where $\gamma = \beta$.

For a regular region $\Omega \supset \bar{D}$ we denote by γ_Ω the part of $\partial\Omega$ which is "homologous" to γ. The remainder $\delta_\Omega = \partial\Omega - \gamma_\Omega$ consists of a finite number of analytic Jordan curves $\delta_{\Omega j}$. For a regular exhaustion $\{R_n\}_{n=0}^\infty$ with $R_0 \supset \bar{D}$ and nonempty γ_{R_0}, set $\gamma_n = \gamma_{R_n}$ and $\delta_{nj} = \delta_{R_n j}$. Then there exists a unique function $p_{\gamma_n} \in H(R_n - \zeta)$ satisfying

(a) $p_{\gamma_n} | D = \log|z - \zeta| + h_n(z)$ with $h_n \in H(\bar{D})$ and $h_n(\zeta) = 0$,

(b) $p_{\gamma_n} | \gamma_n = k_n(\gamma)$ (const.) and $p_{\gamma_n} | \delta_{nj} = d_{nj}$ (const.) so that $\int_{\delta_{nj}} * dp_{\gamma_n} = 0$, which is called a capacity function of γ_n (Sario [6]). It is known that $k_n(\gamma)$ increases with n and the limit $k(\gamma)$ is independent of the choice of $\{R_n\}_{n=0}^\infty$. We call $e^{-k(\gamma)}$ the capacity of γ and denote it by cap γ. When cap $\gamma > 0$, p_{γ_n} converges to a functions p_γ, which is independent of the choice of the exhaustion (Sario [6]). Even when cap $\gamma = 0$, we can also choose a subsequence of $\{p_{\gamma_n}\}$ which converges to a function p_γ. Such functions p_γ will be called capacity functions of γ (Sario [6]). As mentioned above there exists only one capacity function when cap $\gamma > 0$.

It is the purpose of this paper to determine the number of capacity functions p_γ when cap $\gamma = 0$.

2. The harmonic dimension of γ. Let R, β, γ and δ be as in 1. Furthermore we suppose that γ *is of zero capacity*. For a regular region $\Omega \supset \bar{D}$ we denote by $V_{\Omega i}$ components of $R - \bar{\Omega}$ whose derivations are contained in γ and by $W_{\Omega j}$ the remaining components. Here an ideal boundary component will be called a derivation of $V_{\Omega i}$ when it is contained in the closure of $V_{\Omega i}$ in the compactification of R. Here-

after we always choose Ω so large as to make the derivations of $W_\Omega = \bigcup_j W_{\Omega j}$ contain in δ. Therefore W_Ω is always a neighborhood of all of δ.

We consider the normal operator $L_1^{(\Omega)}$ with respect to $R - \bar{\Omega}$ associated with the partition $P = \gamma + \sum_j \delta_j$ of β where δ_j is a component of δ (Ahlfors-Sario [1]).

Let q be a harmonic function in $R - \zeta$. Then q will be called *of L_1-type at δ* when $q = L_1^{(\Omega)}q$ in W_Ω for an admissible Ω. It is easy to see that this property depends only on δ, i.e., if $q = L_1^{(\Omega)}q$ in W_Ω, then $q = L_1^{(\Omega')}q$ in $W_{\Omega'}$ for every admissible Ω'.

We denote by $HP_0(V_\Omega)$ the family of functions u such that u is a positive harmonic function in $V_\Omega = \bigcup_i V_{\Omega i}$ with boundary values zero at $\gamma_\Omega = \partial V_\Omega$. We may extend u to be identically zero in W_Ω. Moreover we consider the following two families of functions. The first family N_Ω consists of $u \in HP_0(V_\Omega)$ such that $\int_{\gamma_\Omega} {}^*du = 2\pi$ where γ_Ω is positively oriented with respect to Ω. The second family is the family F of $q \in H(R - \zeta)$ having the following properties:

(c) $q \mid D = \log|z - \zeta| + h(z)$ with $h \in H(\bar{D})$ and $h(\zeta) = 0$,

(d) q is of L_1-type at δ,

(e) q is bounded from below near γ.

In addition to the obvious fact that N_Ω and F are convex, they are related to each other as follows.

LEMMA. *There exists a bijective map T of N_Ω onto F satisfying*

(f) *$T(\lambda u + (1-\lambda)v) = \lambda Tu + (1-\lambda)Tv$ for $u, v \in N_\Omega$, $0 < \lambda < 1$,*

(g) *$Tu - u$ is bounded in V_Ω.*

For the proof let $u \in N_\Omega$ and denote by L the direct sum of $L_1^{(\Omega)}$ and the Dirichlet operator with respect to D (Sario [5]). Take the singularity function s_u on $(R - \bar{\Omega}) \cup (D - \zeta)$ defined by $s_u = u$ in $R - \bar{\Omega}$ and $s_u = \log|z - \zeta|$ in $D - \zeta$. Since the total flux of s_u is zero, the equation $p - s_u = L(p - s_u)$ has a unique solution p_u on R, up to an additive constant. Normalize p_u so as to satisfy (c) and set $Tu = p_u$. Obviously $Tu \in F$. Since γ is of zero capacity, T is clearly injective. The property in (f) and (g) follows easily from the definition of T.

To see the surjectivity let $q \in F$. We denote by Bq the bounded harmonic function in V_Ω with the boundary values $q \mid \gamma_\Omega$ at γ_Ω. Set $u = q - Bq$ in V_Ω and $u = 0$ in W_Ω. Since q is of L_1-type at δ and bounded from below near γ, $u \in N_\Omega$. Therefore we have only to show that $q - s_u = L(q - s_u)$ in $(R - \bar{\Omega}) \cup (D - \zeta)$. By the definition of u, $q - u = Bq$ in V_Ω and $L_1^{(\Omega)}(q - u) = L_1^{(\Omega)}q$ in V_Ω. Furthermore $Bq - L_1^{(\Omega)}q$ is bounded in V_Ω and vanishes on γ_Ω. Hence $Bq = L_1^{(\Omega)}q$

in V_Ω. On the other hand, $L_1^{(\Omega)}(q-u) = L_1^{(\Omega)}q$ in W_Ω. Consequently $q-u = L(q-u)$ also in W_Ω. Finally it is obvious that the same equality holds in $D-\zeta$.

3. We denote by M_Ω the set of all minimal function in $HP_0(V_\Omega)$ normalized as $\int_{\gamma_\Omega} {}^*du = 2\pi$. Lemma 2 guarantees that the cardinal number of M_Ω is independent of the choice Ω. Extending Heins' definition (Heins [3]), we call it the harmonic dimension of γ, which we shall denote by d_γ.

4. **The number of capacity functions.** We are now able to state our main result:

THEOREM. *Suppose that γ is an isolated closed subset of zero capacity in the ideal boundary of R. If the harmonic dimension of γ is 1, then the capacity function of γ is unique. If the harmonic dimension of γ is greater than 1, there are a continuum of capacity functions of γ.*

Denote by C_γ the family of all capacity functions of γ, by c_γ the cardinal number of C_γ and also by ψ the cardinal number of the continuum. Then the statement of our theorem can also be summarized in a single formula as follows:

$$c_\gamma = 1 + (d_\gamma - 1)\psi \,. \tag{1}$$

5. Before entering the proof we need two lemmas, which will be used to show that $C_\gamma = F$. Let R_n, γ_n and δ_{nj} be as in 1. Set $V_{ni} = V_{R_n i}$ and $W_{nj} = W_{R_n j}$ (see 2). Moreover put $\Omega_{0n} = R - \bar{V}_0 - \bar{W}_n$ with $V_0 = \bigcup_i V_{0i}$ and $W_n = \bigcup_j W_{nj}$.

LEMMA. *Let $p \in F$. Then there exists a sequence $\{p_n\}_{n=0}^\infty$ with $p_n \in H(\Omega_{0n} - \zeta)$ satisfying*

(h) $p_n \mid D = \log|z-\zeta| + h_n(z)$ *with* $h_n \in H(\bar{D})$ *and* $h_n(\zeta) = 0$,

(i) $p_n \mid \gamma_0 = p + k_n$ (const.) *and* $p_n \mid \delta_{nj} = d_{ni}$ (const.) *with*

$$\int_{\delta_{nj}} {}^*dp_n = 0 \,,$$

(j) $\{p_n\}$ *converges uniformly to* p *on any compact* K *with*

$$\bar{K} \subset \Omega_0 = R - \bar{V}_0 - \zeta \,.$$

For the proof construct p_n with (h) and (i) by the linear operator method of Sario [5]. Denote by D_ε a parametric disk about ζ with radius ε and by α_ε its circumference. We orient α_ε and γ_0 negatively with respect to $\Omega_{0n} - \bar{D}_\varepsilon$ and write according to Ahlfors-Sario [1]:

$$A_\varepsilon(p) = \int_{\alpha_\varepsilon+\gamma_0} p^*dp\ , \quad B_n(p) = \int_{\delta_n} p^*dp\ , \quad A_\varepsilon(p, q) = \int_{\alpha_\varepsilon+\gamma_0} p^*dq$$

and

$$B_n(p, q) = \int_{\delta n} p^*dq\ .$$

For $m > n$ we denote by $D_{n,\varepsilon}(p_m - p_n)$ and $D_n(p_m - p_n)$ Dirichlet integrals of $p_m - p_n$ taken over $\Omega_{0n} - \bar{D}_\varepsilon$ and Ω_{0n} respectively. Since $B_n(p_n) = 0,\ B_n(p_n, p_m) = 0,$

$$D_{n,\varepsilon}(p_m - p_n) = B_n(p_m) + 2A_\varepsilon(p_n, p_m) - A_\varepsilon(p_n) - A_\varepsilon(p_m)\ .$$

Observing that $B_n(p_m) < 0$ and letting $\varepsilon \to 0$,

(2) $\quad D_n(p_m - p_n) \leqq a_m - a_n$ where $a_j = \int_{\gamma_0} p^*dp_j + 2\pi k_j \quad (j = n, m)$.

Moreover we construct another sequence $q_n \in H(\Omega_{0n} - \zeta)$ satisfying

(h′) $\quad q_n \mid D = \log \mid z - \zeta \mid + h'_n(z)$ with $h'_n \in H(\bar{D})$ and $h'_n(\zeta) = 0$,

(i′) $\quad q_n \mid \gamma_0 = p + k'_n$ (const.) and the normal derivative of q_n vanishes on δ_n. By the same way as above we obtain

(3) $\quad D_n(q_m - q_n) \leqq b_n - b_m$ where $b_j = \int_{\gamma_0} p^*dq_j + 2\pi k'_j \quad (j = n, m)$

and

(4) $$D_n(p_n - q_n) = b_n - a_n\ .$$

From (2), (3) and (4) we see a_n is increasing and b_n is decresing as n increases and that $a_n \leqq b_n$. Therefore $\lim_n a_n$ and $\lim_n b_n$ exist and are finite. In particular it follows from (2) that p_n converges uniformly to p on any compact K with $\bar{K} \subset \Omega_0$.

6. The following lemma is easy to see and plays an important role in the proof of our theorem.

LEMMA. *Let $p \in F$. Then there exist an exhaustion $\{R_n\}_{n=0}^\infty$ and a sequence $\{p_n\}_{n=0}^\infty$ with $p_n \in H(R_n - \zeta)$ having the properties* (h) *of Lemma* 5 *and*

(k) $\quad p_n \mid \gamma_n = p + k_n$ (const.) *and* $p_n \mid \delta_{nj} = d_{nj}$ (const.) *with*

$$\int_{\delta_{nj}} {}^*dp_n = 0\ ,$$

(l) $\quad \{p_n\}$ *converges uniformly to p on any compact K in $R - \zeta$.*

Since γ has zero capacity we can see that there exists an Evans potential e_0 for γ, i.e., a function $e_0 \in H(R-\zeta)$ satisfying the following conditions (Nakai [4]):

(m) $e_0 \mid D = \log|z-\zeta| + w(z)$ with $w \in H(\bar{D})$ and $w(\zeta)=0$,

(n) e_0 is of L_1-type at δ,

(o) $\lim_{z\to\gamma} e_0(z) = +\infty$.

Needless to say $e_0 \in F$.

7. **Proof of theorem.** Consider $p_\lambda = \lambda e_0 + (1-\lambda)q$ with a fixed $q \in F$ and $0<\lambda<1$. It is clear that $\lim_{z\to\gamma} p_\lambda(z) = +\infty$ and $p_\lambda \in F$. Therefore by Lemma 6 we obtain

$$\{p_\lambda\}_{0<\lambda<1} \subset C_\gamma \, . \tag{5}$$

On the other hand, obviously

$$C_\gamma \subset F \, . \tag{6}$$

Moreover observe that $\lambda \longrightarrow p_\lambda$ is injective if $e_0 \neq q$.

By the approximation theorem of Heins [2], we can see at once that if $d_\gamma = 1$, so is the cardinal number of F. It is trivial that the converse is valid. Hence $c_\gamma = 1$ if and only if $d_\gamma = 1$.

Suppose that $d_\gamma \geqq 2$. Then there exists a $q \in F$ with $q \neq e_0$. By the injectivity of $\lambda \longrightarrow p_\lambda$, $\psi \leqq c_\gamma$. Conversely it follows from (6) that $c_\gamma \leqq$ the cardinal number of F which is not greater than ψ. Thus $c_\gamma = \psi$. In either case, since $d_\gamma \leqq \psi$, we have $c_\gamma = 1 + (d_\gamma - 1)\psi$.

The author would like to express his warmest thanks to Professor Nakai for his kind guidance. He is also grateful for the valuable comments of the refree.

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Received October 2, 1967 and in revised form February 27, 1968. This is a part of the author's thesis for the partial satisfaction of the degree Master of Science at Nagoya University.

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