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REARRANGEMENT OF SPHERICAL MODIFICATIONS

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A "rearrangement" theorem of Wallace states essentially that if a manifold M is the trace of a sequence of spherical modifications of various types then these modifications can be arranged so that the order in which they are performed is that of increasing type, their trace still being M. In this paper a related rearrangement problem is considered; namely, to determine bounds on how "mixed" the order of performing a sequence of modifications can be and still possess the same trace M.

Sections 2 through 4 are concerned with basic definitions and preliminary results. The most important of which is the establishment of an algorithm to determine a measure of how "mixed" the order of a sequence of integers is [§ 4]. The main results appear in § 5.

2. Spherical modifications.

Unless stated otherwise an n-manifold is a compact, differentiable n-dimensional manifold without boundary.

Let V_1 be an n-manifold and suppose S^i is an i-sphere homeomorphically and smoothly imbedded in V_1 with a trivial normal bundle. Then S^i has a neighborhood of the form $S^i \times D^{n-i}$. (D^{n-i} is an $(n-i)$ – disc). Clearly the boundary of $S^i \times D^{n-i} = S^i \times S^{n-i-1} =$ the boundary of $D^{i+1} \times S^{n-i-1}$. Smoothly identifying the boundary of $D^{i+1} \times S^{n-i-1}$ with the boundary of (V_1-interior $(S^i \times D^{n-i})$) results in a new manifold V_2. V_2 is said to be obtained from V_1 by a spherical modification of type i, or by an i-type modification. ([7], p. 504).

Associated to the spherical modification is an $n+1$-manifold W called the trace of the modification. The boundary of $W = V_1 \cup V_2$ and the triple $(W; V_1, V_2)$ is a manifold triad in the sense of [4] page 2. As a matter of convention, performing a type-1 modification on V_1 is taken to mean $V_1 = \varnothing$, $V_2 =$ an n-sphere, and the trace is an $n+1$ – disc. The n will be clear from context. For a further discussion of the trace see [8] page 775.

3. Realizable sequences.

DEFINITION 3.1. An admissible sequence $S(n)$ is a finite sequence of integers $a_1, a_2, \cdots, a_l$ with $a_1 = -1, a_l = n-1, -1 \leqq a_i \leqq n-1$ for $i = 1, 2, \cdots, l$ and $a_i \neq a_j$ if $i \neq j$.

DEFINITION 3.2. Let $S(n)$ be an admissible sequence. $S(n)$ is realizable by an n-manifold M if M is diffeomorphic to the combined trace of a sequence of spherical modifications of type $a_1, a_2, \cdots, a_l$. More explicitly, there exists a sequence of $(n-1)$-manifolds $V_1, V_2, \cdots, V_{l+1}$ where V_{i+1} can be obtained from V_i by performing a finite number of type a_i modifications on V_i and the combined trace of these modifications is diffeomorphic to M.

Note that $S(n)$ only tells the type of modifications that appear and gives no information on how many modifications of the various types occur. Also it is only necessary to consider admissible sequences since M is assumed to be without boundary and compact.

When no confusion will arise the n in $S(n)$ will be deleted.

The following lemmas will be used in proving the results of § 5.

LEMMA 3.1. *Let $S = \{a_1, \cdots, a_i, a_{i+1}, \cdots, a_j, \cdots, a_l\}$ be an admissible sequence realizable by M. If all the integers following a_i and preceding a_j in S are larger than a_j then the sequence $S' = \{a_1, \cdots, a_i, a_j, a_{i+1}, \cdots, a_l\}$ is also realizable by M.*

Proof. See § 4 of [7] or § 4 of [4].

LEMMA 3.2. *Let $S=\{a_1, \cdots, a_i, a_{i+1}, \cdots, a_{i+k}, \cdots, a_l\}$ be an admissible sequence realizable by M. If $a_{i+j} > a_{i+j+1}$ for $j = 0, 1, \cdots, k-1$ then M is diffeomorphic to the trace of a sequence of modifications with all modifications of type a_{i+j} $(j = 0, 1, \cdots, k)$ performed on a single manifold. More explicitly, there exists a sequence of manifolds $V_1, \cdots, V_i, V_{i+k+1}, V_{i+k+2}, \cdots, V_{l+1}$ where V_{i+k+1} can be obtained from V_i by modifications of type a_{i+j} $(j = 0, 1, \cdots, k)$ (the a_{i+j}-spheres $j = 0, 1, \cdots, k$ determining these modifications can be assumed mutually disjointly imbedded in V_i). Further V_{j+1} can be obtained from V_j by a_j type modifications for $i \neq j \neq i+k$ and the combined trace of all the modifications is diffeomorphic to M.*

Proof. See § 4 of [4], § 4 of [7], or [9].

THEOREM 3.1. (*Rearrangement Theorem*). *Let S be an admissible sequence realizable by M. Then the sequence $S' = \{a_1, \cdots, a_l\}$ composed of the same integers as S and satisfying $a_i < a_{i+1}$ for $i = 1, 2, \cdots, l-1$ is also realizable by M.*

Proof. This follows easily by repeated application of Lemma 3.1.

Theorem 3.1 also follows from a more general rearrangement theorem due to Smale [6] and Wallace [7]. Also see ([4], p. 44, Th. 4.8).

The above theorem asserts that any sequence of modifications realizable by M can still be realized by M when those modifications are performed in order of increasing type. The purpose of this paper is to study how "mixed" the order of the modifications may be and still be realizable by M. To this end a study of sequences will now be made.

4. Shifts on sequences.

DEFINITION 4.1. Let $S = \{a_i\}$ $i = 1, 2, \cdots, l$ be a finite sequence of distinct integers. A shift of S consists of removing some a_j from S and replacing it in $S - \{a_j\}$ in a position different from its original position.

DEFINITION 4.2. $d(S)$ = minimum number of shifts needed to put S in natural order.

A method of computing $d(S)$ which will be useful in proving the main results of this paper will now be described.

DEFINITION 4.3. The full decreasing subsequence $S_1 = \{a_{1j}\}$ $j = 1, \cdots, n_1$ of S is the subsequence of S obtained as follows: a_{11} is the first integer appearing in S, a_{12} is the first integer following a_{11} in S satisfying $a_{12} < a_{11}$, a_{13} is the first integer following a_{12} in S satisfying $a_{13} < a_{12}$ etc. After a finite number of such steps the process must terminate with the last element so determined being a_{1n_1}.

DEFINITION 4.4. The number of decreasing subsequences of S is the smallest integer m satisfying $(S - \bigcup_{i=1}^{m} S_i) = \varnothing$ where $S_i = \{a_{ij}\}$ $j = 1, \cdots, n_i$ is the full decreasing subsequence of $(S - \bigcup_{j=1}^{i-1} S_j) = S$ with the subsequences $S_1, S_2, \cdots, S_{i-1}$ deleted.

THEOREM 4.1. *Let S be a sequence composed of l distinct integers. If S has m decreasing subsequences then $d(S) = l - m$.*

Proof. The proof will be by induction on the length l *of* S. If $l = 1$ the theorem clearly holds.

Assume now that the theorem is valid for sequences of length $< l$. Let S have length l and m decreasing subsequences with n_i integers in the i^{th} subsequence S_i.

If $d(S) = 0$ it is clear that $n_i = 1$ for $i = 1, \cdots, m$. Thus $l = m$ and $d(S) = l - m$.

Suppose then that $d(S) \neq 0$. Under this assumption there is defined a unique integer p satisfying $n_p > 1$ and if $p < r \leqq m$ then $n_r = 1$. Let S' be S with a_{pn_p} deleted if $p = m$ or if $p \neq m$ and

$a_{p1} < a_{p+1,1}$, and with a_{p1} deleted if $p \neq m$ and $a_{p+1,1} < a_{p1}$.

LEMMA 4.1. $d(S) \leqq d(S') + 1$.

Proof. One can imagine the $d(S')$ shifts used on S' to put it in natural order as being performed on S (simply ignore a_{p1} (or a_{pn_p})). Hence after the $d(S')$ shifts on S one obtains a sequence S'' in which all integers are in natural order except for possibly one, a_{p1} (or a_{pn_p}). At most, then, one more shift will put S'' in natural order. This proves the lemma.

LEMMA 4.2. $d(S') = l - 1 - m$.

Proof. Since the length of S' is $l - 1$ the induction hypothesis implies $d(S') = l - 1 - m'$ where m' = number of decreasing subsequences of S'. It must be shown that $m = m'$.

Let $A = \{S_1, \cdots, S_m\}$ and $A' = \{S'_1, \cdots, S'_{m'}\}$ be the sets of full decreasing subsequences of S and S' respectively. Clearly deleting an element from the p^{th} sequence S_p does not change any of the preceding sequences. Hence A and A' are identical up to the p^{th} sequence. Consider now the p^{th} sequence.

Case 1. $p = m$ or if $p \neq m$ then $a_{p1} < a_{p+1,1}$. This implies that a_{p1} is not deleted and thus sequence p of both A and A' begins with a_{p1}. It will now be shown that $S_i = S'_i$ for $p < i \leqq m$. For suppose that some a_{i1} $(i > p)$ appears in S'_p. The existence of such an a_{i1} implies that $p \neq m$, thus $a_{p1} < a_{p+1,1}$. By construction $a_{p+1,1} < a_{p+2,1} < \cdots < a_{m1}$ and consequently $a_{p1} < a_{i1}$. But also by construction a_{p1} is the largest integer of S'_p, hence no a_{i1} $(p < i \leqq m)$ can appear in S'_p. From the relation $a_{i1} < a_{i+1,1}$ $(i = p + 1, \cdots, m - 1)$ it easily follows now that each a_{i1} $(p < i \leqq m)$ constitutes a sequence of A'. Therefore $m = m'$.

Case 2. $p \neq m$ and $a_{p+1,1} < a_{p1}$. This implies that a_{p1} is deleted. Suppose now that a_{i1} $(i > p)$ appears in S'_p. By definition, a_{i1} appears in S after a_{p1}. Let a_{pj} be the smallest integer in S_p that precedes a_{i1} in S. Then $a_{pj} < a_{i1}$ or a_{i1} would be in S_p. Furthermore a_{pj} cannot appear in S'_p, since a_{i1} does. But the only way a_{pj} could fail to be in that sequence is by being deleted; that is, $j = 1$. However since $a_{p+1,1} < a_{p1}$ and $a_{p+1,1}$ follows a_{p1} in S there must be an element of the form a_{pk} $(k \neq 1)$ preceding $a_{p+1,1}$. Since a_{i1} is $a_{p+1,1}$ or follows it in S, a_{i1} must also follow a_{pk} $(k \neq 1)$. Hence $a_{pj} \neq a_{p1}$ which is a contradiction.

The same argument as in Case 1 now shows that each a_{i1} $(p < i \leqq m)$ constitutes a sequence of A'. Hence $m = m'$.

In all cases, then, $m = m'$ which proves the lemma.

LEMMA 4.3. $d(S) \geqq l - m$.

Proof. Consider $S_i = \{a_{ij}\}$ $j = 1, \cdots, n_i$. These elements appear in S in the given order. By construction $a_{in_i} < \cdots < a_{i2} < a_{i1}$, hence it clearly takes at least $n_i - 1$ shifts to put S_i in natural order. Further any shift involving an element of S_i does not affect the relative order of the elements of $S_j (i \neq j)$. Thus the number of shifts needed is $\geqq \sum_{i=1}^{m} (n_i - 1) = (\sum_{i=1}^{m} n_i) - m = l - m$ and the lemma is proved.

Combining Lemma 4.1 and 4.2 gives $d(S) \leqq l - m$ and in view of Lemma 4.3 $d(S) = l - m$. This completes the proof of Theorem 4.1.

REMARK. It is clear that the "natural order" of S is not essential to the above arguments and the results of this section can be generalized to the situation of finding the number of shifts needed to put S in a given order, not necessarily the natural order. Theorem 4.1 still holds in this broader sense and d becomes a distance function on the set of all orderings of S.

5. The main result. Denote by $C(M)$ the strong category of M (cf [1]).

THEOREM 5.1. *Let S be an admissible sequence of length l. If S is realizable by an n-manifold $M (n > 0)$ then* $2 \leqq d(S) + C(M) \leqq l \leqq n + 1$.

Proof. Since M is an n-manifold the only modifications that can occur are of type $-1, 0, 1, \cdots, n - 1$, thus $l \leqq n + 1$. A simple cohomological argument shows $C(M) \geqq 2$ and thus

$$2 \leqq C(M) \leqq d(S) + C(M) .$$

It remains then to establish $d(S) + C(M) \leqq l$, which is equivalent to $C(M) \leqq l - d(S)$. In view of Theorem 4.1, then, Theorem 5.1 will follow if it is shown that $C(M) \leqq m$.

LEMMA 5.1. *If S is an admissible sequence realizable by M then $C(M) \leqq m$ where m is the number of decreasing subsequences of S.*

Proof. Let $A = \{S_1, S_2, \cdots, S_m\}$ be the set of decreasing subsequences of S. Since S is admissible $n_1 = n_m = 1$, $a_{11} = -1$ and $a_{m1} = n - 1$. Let S' be the sequence given in A; that is,

$$S' = a_{11}, a_{21}, a_{22}, \cdots, a_{2n_2}, \cdots, a_{m-1,1}, \cdots, a_{m-1,n_{m-1}}, a_{m1} .$$

It will first be shown that S' is realizable by M. Since $a_{11} = -1$

it follows that all admissible sequences composed of the same integers as S begin with a_{11}. By definition all the integers between a_{2i} and $a_{2,i+1}$ in S are larger than $a_{2,i+1}$ for $i = 1, 2, \cdots, n_2$. Thus by repeated application of Lemma 3.1 the sequence $S_1' = a_{11}, a_{21}, a_{22}, \cdots, a_{2n_2}, \cdots$ is realizable by M where the last $\cdots$ represent the sequence S with a_{11}, a_{2i} $(i = 1, 2, \cdots, n_2)$ deleted. Repeating the argument using sequences $j (j = 3, \cdots, m)$ of A one easily obtains the result.

Now by repeated application of Lemma 3.2 there exists a sequence of $(n-1)$-manifolds $V_1, V_2, \cdots, V_{m+1}$ where V_{i+1} can be obtained from V_i by modifications of type $a_{ij} (j = 1, 2, \cdots, n_i)$ and the combined trace of all these modifications is diffeomorphic to M. A detailed study of the trace will now be made.

Since M is connected the number of -1 type modifications needed can always be assumed to be 1 ([7], p. 518). Thus

$$V_1 = \varnothing, \; V_2 = S^{n-1}$$

and the trace T_1 of this modifications is an n-disc D_1 with $\partial D_1 = V_2 = S^{n-1}$ ("∂" denotes boundary). V_3 is obtained from V_2 by a_{2i} $(i = 1, 2, \cdots, n_2)$ type modifications. Denote by $N(a_{2i})$ the number of a_{2i} type modifications involved. Then $N_2 = \sum_{i=1}^{n_2} N(a_{2i})$ is the number of modifications used in going from V_2 to V_3. Each a_{2i} type modification is determined by an a_{2i}-sphere imbedded in V_2 possessing a normal neighborhood of the form $S^{a2i} \times D^{n-a2i-1}$. Thus there is imbedded in $V_2 N_2$ such spheres having mutually disjoint trivial normal neighborhoods.

Let T_2 be the trace of these N_2 modifications. Then $T_1 \cup T_2$ is homeomorphic to the topological space X constructed as follows: Any n-disc D^n is of the form $D^j \times D^{n-j}$, thus $\partial D^n = S^{j-1} \times D^{n-j} \cup D^j \times S^{n-j-1}$. Attach N_2 different n-discs to V_2 by identifying $S^{ai} \times D^{n-ai-1} \subset \partial D^n$ (let $j = a_i + 1$) with the normal neighborhood of the a_i-sphere, the attaching diffeomorphisms being determined by the spherical modifications involved. Thus X is D_1 with N_2 n-discs attached to ∂D_1. Hence $T_1 \cup T_2 = D_1 \cup D_2$ where D_1 is an n-disc and D_2 is the union of N_2 mutually disjoint n-discs. For a detailed discussion of this description of the trace see § 3 of either [5] or [8].

Repeating this argument through V_{m+1} one has that $M = \bigcup_{i=1}^{m} T_i = \bigcup_{i=1}^{m} D_i$ where D_i is the union of $N_i = \sum_{j=1}^{n_i} N(a_{ij})$ mutually disjoint n-discs. If $(C_1, C_2, \cdots, C_k)$ is a set of mutually disjoint n-discs in a connected n-manifold (which M is) then C_1 can be joined to C_2 by a smooth arc α so that $\alpha \cap \bigcup_{i=1}^{k} C_i =$ two points, one in ∂C_1 and one in ∂C_2. Thus C_1 and C_2 can be joined to form a set E_2 which is contractible in itself. Repeating this construction on $(E_2, C_3, \cdots, C_k)$ starting with E_2 and C_3 to form E_3 one obtains a set E_k which is contractible in itself.

Hence M can be covered by m such contractible sets and if each of them is slightly "blown up" then M can be covered by m contractible in themselves, open sets. Thus $C(M) \leqq m$. This completes the proof of Lemma 5.1 and thus of Theorem 5.1.

COROLLARY 5.1. *If an admissible sequence S of length l is realizable by M and $l = C(M)$ then $d(S) = 0$. In particular if $C(M) = n + 1$ then any admissible sequence realizable by M must satisfy $d(S) = 0$.*

Proof. The corollary follows directly from Theorem 5.1.

There are many manifolds M satisfying $C(M) = n + 1$. For example $C(P^n) = n + 1$ where P^n is n-dimensional real projective space. If n is even, all connected manifolds M in the cobordism class of P^n satisfy $C(M) = n + 1$. Actually it can be shown that every cobordism class contains manifolds satisfying this condition and for even dimensional manifolds at least half of all cobordism classes contain only manifolds M satisfying $C(M) = n + 1$ if M is connected [3].

If l is the length of an admissible sequence S realizable by M then Theorem 5.1 asserts that $d(S) + C(M) \leqq l$ and since $C'(M) \leqq C(M)$ ($C'(M)$ is the category of M) it follows that $d(S) + C'(M) \leqq l$. One question that arises is when can an S be found so that equality holds, i.e., $d(S) + C(M) = l$. A connection between this question and the Poincare conjecture will now be given.

COROLLARY 5.2. *Let M be an n-manifold which is a homotopy sphere. If there exists an admissible sequence S of length l realizable by M satisfying $d(S) + C'(M) = l$ then M is homeomorphic to the n-sphere S^n.*

Proof. Since M is a locally contractible, compact, finite dimensional metric space it is an absolute neighborhood retract ([2], p. 32). It is assumed that the homotopy groups vanish in all positive dimensions $\leqq n - 1$ thus by Theorem 18.2 of [1] $C'(M) \leqq 2$ and consequently $C'(M) = 2$. Therefore $d(S) = l - 2$. $d(S) = l - m$ implies then that $m = 2$. But $l = \sum_{i=1}^{m} n_i = \sum_{i=1}^{2} n_i = n_1 + n_2 = 2$ (recall that $n_1 = n_m = 1$); that is, S is the sequence $(-1, n - 1)$, and since M is connected it can be assumed that there is only one modification of each type ([7], Th. 4). In terms of functions this means there exists a Morse function on M with exactly two nondegenerate critical points. (cf. [4], p. 30) Applying Reeb's theorem ([5], p. 25) one has that M is homeomorphic to S^n. This completes the proof.

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