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GAP SERIES AND AN EXAMPLE TO MALLIAVIN'S THEOREM

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0. Malliavin's celebrated theorem of spectral nonsynthesis is based on a real function f of class A

$$f(t) = \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt ,$$
$$\sum |a_n| + \sum |b_n| < \infty ,$$

for which $\int_{-\infty}^{\infty} |u| \, || e^{iuf} ||_\infty \, du < \infty.$

Here and in general $|| g ||_\infty \equiv \sup_n | \hat{g}(n) |$. **This note presents a method for constructing a function f, based on a gap property and a method of estimation of Kahane.**

Let $0 < n_1 < n_2 < \cdots < n_k < \cdots$ be a sequence of integers with the property:

Whenever $\varepsilon_k = 0, \pm 1$, and $\varepsilon_1 n_1 + \cdots + \varepsilon_N n_N = 0$, then $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_N = 0$.

Let $\omega_1, \omega_2, \cdots, \omega_k, \cdots$ be independent random variables defined upon a probability space Ω, distributed uniformly upon $[0, 2\pi]$. For a number $0 < b < 1$ set

$$f(t) = \sum_{k=1}^{\infty} b^k \cos (n_k t + \omega_k) .$$

Then, for each integer $M \geqq 1$ there is a $b = b(M) < 1$ such that

$$(1) \qquad \int_{-\infty}^{\infty} |u|^M || e^{iuf} ||_\infty \, du < \infty \quad \text{for almost all } \omega \text{ in } \Omega .$$

REMARKS. Choosing $n_k = 2^k$, we obtain a function f of class $\mathrm{Lip}\,(-\log b/\log 2)$, and this shows that $b(M)$ must converge to 1 as $M \to \infty$. For if the integral in (1) is finite, there is a number ξ such that $(f - \xi)^M$ does not admit synthesis, and it must be false that

$$| f(t) - \xi |^{2M} = O(d(t, f^{-1}(\xi))) ,$$

[3, pp. 116, 122]. But then $f \notin \mathrm{Lip}\,(2^{-1/M})$. Functions f with the Lipschitz condition were first produced in [1], and an explicit example—that is, nonprobabilistic—given in [2].

1. Let $0 < r < 1$, $0 < \varepsilon$, $0 < \eta < (1 - r) \log 5 - \log 4$. Define $B_N(s, t)$ for $0 < s, t < 2\pi (N = 1, 2, 3, \cdots)$ to be the number of integers k defined by

$$1 \leqq k \leqq N , \qquad |\cos n_k s - \cos n_k t| \geqq \varepsilon .$$

LEMMA. *If $\varepsilon > 0$ is small enough, the Lebesgue measure*

$$m\{B_N(s, t) \leqq rN\} = O(e^{-\eta M}), \quad as \quad N \to \infty .$$

Proof. Set

$$\xi_k(s, t) = 5 - (\cos n_k s - \cos n_k t)^2$$

or

$$\xi_k = 4 - \tfrac{1}{2} \cos 2n_k s + 2 \cos n_k s \cos n_k t - \tfrac{1}{2} \cos 2n_k t .$$

The mean of the product $\xi_1 \cdots \xi_N$ is 4^N. For the product is a sum of terms

$$c\, \Pi' \cos 2n_k s\, \Pi'' \cos n_k s \cos n_k t\, \Pi''' \cos 2n_k t ,$$

where the symbols Π', etc., refer to products over mutually disjoint subsets of $\{1, 2, \cdots, N\}$. If such a sum has mean $\neq 0$, it is trivial, for there are integers $\varepsilon_k = \pm 1$, $\delta_k = \pm 1$, defined for every exponent n_k present, such that $2\Sigma'\varepsilon_k n_k + \Sigma''\varepsilon_k n_k = \Sigma''\delta_k n_k + 2\Sigma'''\delta_k n_k = 0$. But $\Sigma'\varepsilon_k n_k + \frac{1}{2}\Sigma''(\varepsilon_k + \delta_k)n_k + \Sigma'''\delta_k n_k = 0$, where $\frac{1}{2}(\varepsilon_k - \delta_k) = 0, \pm 1$. Thus Π' and Π''' must be trivial, and so finally Π'' is trivial.

Now

$$\{B_N \leqq rN\} \subseteqq \{\xi_1 \cdots \xi_N \geqq (5 - \varepsilon^2)^{(1-r)N}\} ,$$

so

$$m\{B_N \leqq rN\} \leqq 4\pi^2[4/(5 - \varepsilon^2)^{1-r}]^N ,$$

and we need only choose $\varepsilon > 0$ so that $\eta < (1 - r) \log (5 - \varepsilon^2) - \log 4$. We now choose $\varepsilon > 0$, $\eta > 0$, $1 > r > 0$, once and for all.

2. Following [1] we observe that for g in L^2

$$g(t) = \sum_{-\infty}^{\infty} c_n e^{int}$$

$$(g * g)(t) = (2\pi)^{-1} \int g(t - s)g(s)ds = \sum_{-\infty}^{\infty} c_n^2 e^{int}$$

$$||g * g||_2^2 = (2\pi)^{-1} \iiint g(t - s)g(s)\overline{g(t - p)}\overline{g(p)}dsdtdp = \sum_{-\infty}^{\infty} |c_n|^4 \geqq ||g||_\infty^4 .$$

Set

$$P(x, y, z, \omega) = \cos (x - y + \omega) + \cos (y + \omega) - \cos (x - z + \omega) - \cos(z + \omega) .$$

For fixed x, y, z, P is a trigonometric monomial in ω, say $\tau \sin(\omega + c)$, and τ can be estimated by setting

$$z' = z - \tfrac{1}{2}x\,, \qquad y' = y - \tfrac{1}{2}x\,.$$

We find that $\tau^2 = 4|\cos z' - \cos y'|^2$. Now

$$\begin{aligned}&\exp iu[f(t-s) + f(s) - f(t-p) - f(p)]\\ &\quad = \exp iu \sum_{k=1}^{\infty} b^k P(n_k t, n_k s, n_k p, \omega_k)\,.\end{aligned}$$

To obtain an upper bound for the expectation of $\|e^{iuf}\|_\infty^4$ we integrate this formula, first with respect to $\omega_1, \omega_2, \cdots$ and then with respect to s, p, t. Note the estimation

$$J_0(R) = (2\pi)^{-1}\int_0^{2\pi} e^{iR\sin\omega}d\omega \leqq C(1 + |R|)^{-1/2}\,, \qquad -\infty < R < \infty\,.$$

$$\begin{aligned}&(2\pi)^{-3}\iiint \prod_{k=1}^{\infty} |J_0(2ub_k \cdot |\cos n_k y' - \cos n_k z'|)|\,dxdydz\\ &\quad \leqq (2\pi)^{-2}\iint \prod_{1}^{N(u)} |J_0(2ub^k \cdot |\cos n_k y - \cos n_k z|)|\,dydz\,.\end{aligned}$$

Here $N(u)$ is the integral part of $-\frac{1}{2}\log u/\log b$. If $B_{N(u)}(y, z) \geqq rN(u)$ the product in the integral is at most $(C'|u|^{-1/4})^{rN(u)}$, a magnitude ultimately smaller than any assigned power of $|u|^{-1}$. The integral on the complement $\{B_{N(u)} \leqq rN(u)$ is $O(e^{-\eta N(u)}) = O(|u|^{2^{-1}\eta/\log b})$. Choosing b close to 1, we can make this $O(|u|^{-4M-6})$. Then by Fubini's theorem

$$E\left(\int_{-\infty}^{\infty} |u|^{4M+4}\|e^{iuf}\|_\infty^4 du\right) = \int_{-\infty}^{\infty} |u|^{4M+4}E(\|e^{iuf}\|_\infty^4)du < \infty\,,$$

so $\int_{-\infty}^{\infty} |u|^{4M+4}\|e^{iuf}\|_\infty^4 du < \infty$ for almost all ω in Ω. Conclusion (1) is a consequence of Holder's inequality.

It is clear that if b^k is replaced by k^{-2} for example, the condition (1) is valid for any integer M.

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