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ON THE DECOMPOSITION OF INFINITELY DIVISIBLE PROBABILITY LAWS WITHOUT NORMAL FACTOR

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In the theory of the decomposition of probability laws, the fundamental problem stated by D. A. Raikov of the characterization of the class I_0 of the infinitely divisible laws without indecomposable factors has been studied in the case of univariate laws by Yu. V. Linnik and I. V. Ostrovskiy. Lately, we have shown that nearly all these results can be extended to the case of multivariate laws. In this paper, we give a result which can be considered as an extension of a theorem of Raikov and P. Lévy and of a particular case of theorems of Linnik, and the extension of this result to the case of several variables.

If we consider the finite products of Poisson laws, i.e., the characteristic functions of the variable t of the form

$$f(t) = \exp\left\{ict + \sum_{j=1}^{p} \lambda_j[\exp(i\alpha_j t) - 1]\right\}$$

(c real, $\lambda_j > 0$, $\alpha_j > 0$), three general results are known, the first being owed to D. A. Raikov [9] and P. Lévy [4] and the third to Yu. V. Linnik [5, Chapter 9]:

(a) if $\alpha_1, \cdots, \alpha_p$ are rationally independent, f has no indecomposable factor;

(b) if $\alpha_1, \cdots, \alpha_p$ are such that $0 < a \leqq \alpha_j \leqq 2a$ $(j = 1, \cdots, p)$, f has no indecomposable factor;

(c) if α_{j+1}/α_j is an integer greater than 1 $(j = 1, \cdots, p-1)$, f has no indecomposable factor.

Lately, I. V. Ostrovskiy [8] has extended the two results (a) and (b) of Raikov and Lévy to the case of a continuous spectrum, the base of his study being the

THEOREM 1. (see also [1] chapter 8). *Let f_0 be the infinitely divisible characteristic function of the variable t defined by*

$$f_0(t) = \exp\{i\gamma t + \int_a^b [\exp(ixt) - 1]d\mu(x)\},$$

where γ is a real constant and μ is a nonnegative measure defined on the segment $[a, b]$ $(0 < a < b < \infty)$. If f_1 is a factor of f_0, then

$$f_1(t) = \exp\left\{ict + \int_a^b [\exp(ixt) - 1]dm(x)\right\},$$

where c is a real constant and m is a measure defined on the segment $[a, b]$ *which is nonnegative on* $[a, 2a[$. *Moreover,*

$$S(m) \subset [a, b] \cap (\infty S(\mu)) ,$$

where $S(N)$ *means the support of a measure N and* (∞A) *is defined by*

$$(1)A = A ; \qquad (p)A = (p-1)A + A ; \qquad (\infty A) = \bigcup_{p=1}^{\infty} (p)A$$

(the symbol $+$ indicates the vectorial sum of two subsets of R).

He gives also a more general result which can be stated in the following manner:

THEOREM 2. *Let* f_0 *be the infinitely divisible characteristic function of the variable t defined by*

$$f_0(t) = \exp\left\{i\gamma t + \int_a^b [\exp(ixt) - 1] d\mu(x) + \sum_{k=1}^{\infty} \lambda_k [\exp(i\alpha_k t) - 1]\right\} ,$$

where $\gamma \in R$, $\lambda_k \geqq 0$, $\alpha_k > 0$ $(k = 1, 2, \cdots)$ *and where the following conditions are satisfied*:

(1) *the measure* μ *is a nonnegative measure defined on the segment* $[a, b]$ $(0 < a < b < \infty)$;

(2) *there exists a positive constant K such that*

$$\lambda_k = 0[\exp(-K\alpha_k^2)] \qquad (k \to +\infty) ;$$

(3) $\alpha_1 > b$ *and* α_{k+1}/α_k *is an integer greater than* 1 $(k = 1, 2, \cdots)$. *If* f_1 *is a factor of* f_0, *then*

$$f_1(t) = \exp\left\{ict + \int_a^b [\exp(ixt) - 1] dm(x) + \sum_{k=1}^{\infty} l_k [exp(i\alpha_k t) - 1]\right\} ,$$

where c is a real constant and the following conditions are satisfied:

(a) $$0 \leqq l_k \leqq \lambda_k \qquad (k = 1, 2, \cdots) ;$$

(b) *the measure m is a measure defined on the segment* $[a, b]$ *which is nonnegative on* $[a, 2a[$ *and such that*

$$m(\{b\}) \geqq 0 , \qquad S(m) \subset [a, b] \cap (\infty S(\mu)) .$$

Using the Theorems 1 and 2, we give in § 2, two theorems which can be considered as extensions of the results (a) and (c) stated above. Using the auxiliary results stated in § 3, we extend these results to the case of several variables in the § 4.

2. The case of one variable.

THEOREM 3. *Let* f_0 *be the infinitely divisible characteristic* *function of the variable t defined by*

$$f_0(t) = \exp\left\{i\gamma t + \sum_{j=1}^{p}\sum_{k=1}^{r_j} \lambda_{j,k}[\exp(i\alpha_{j,k}t) - 1]\right\},$$

where γ is a real constant, the $\lambda_{j,k}$ are nonnegative constants and the $\alpha_{j,k}$ are positive numbers satisfying the two conditions

(a) $\alpha_{j,k+1}/\alpha_{j,k}$ *is an integer greater than* 1 $(k = 1, \cdots, r_j - 1; j = 1, \cdots, p)$;

(b) $\alpha_{1,1}, \cdots, \alpha_{p,1}$ *are rationally independent. If f_1 is a factor of f_0, then*

$$f_1(t) = \exp\left\{ict + \sum_{j=1}^{p}\sum_{k=1}^{r_j} l_{j,k}[\exp(i\alpha_{j,k}t) - 1]\right\},$$

where c is a real constant and the $l_{j,k}$ satisfy

$$0 \leqq l_{j,k} \leqq \lambda_{j,k} .$$

Proof. Let f_1 and f_2 be the two characteristic functions such that for any real t

$$f_0(t) = f_1(t)f_2(t) . \tag{2.1}$$

Since f_0 is an entire characteristic function, from Raikov's theorem ([6], theorem 8.1.1), f_j $(j = 1, 2)$ is also entire and the equation (2.1) is also valid for any complex t. Moreover, we have the ridge property ([6], Theorem 7.1.2) which can be written, since f_j is evidently without zeros,

$$u_j(0, y) - u_j(x, y) \geqq 0 \qquad (j = 1, 2) \tag{2.2}$$

for any real x and y where u_j is defined by

$$u_j(x, y) = \operatorname{Re}\log f_j(x + iy) .$$

From the Theorem 1 of the introduction, it follows that for any complex t

$$f_1(t) = \exp\left\{ict + \sum_{j=1}^{p}\sum_{k=1}^{s_j} l'_{j,k}[\exp(ik\alpha_{j,1}t) - 1]\right\},\tag{2.3}$$

where c and the $l'_{j,k}$ are real constants and where s_j is defined by

$$s_j\alpha_{j,1} \leqq \sup_k \alpha_{k,r_k} < (s_j + 1)\alpha_{j,1} .$$

From (2.3), it follows by an elementary computation that

$$u_1(0, y) - u_1(x, y) = 2\sum_{j=1}^{p}\sum_{k=1}^{s_j} l'_{j,k}\sin^2(\tfrac{1}{2}k\alpha_{j,1}x)\exp(k\alpha_{j,1}y) .\tag{2.4}$$

We show now by induction that all the $l'_{j,k}$ for $k\alpha_{j,1} \notin \{\alpha_{j,k}\}$ are equal to zero and that all the $l'_{j,k}$ for $k\alpha_{j,1} \in \{\alpha_{j,k}\}$ are nonnegative. (It is sufficient to show that all the $l'_{j,k}$ are nonnegative since if $l'_{j,k}$ for $k\alpha_{j,1} \notin \{\alpha_{j,k}\}$ is nonnegative, the corresponding term in f_2 is also nonnegative and their sum is zero).

First of all, we show that

$$l'_{j,s_j} \geqq 0 . \tag{2.5}$$

Indeed, from Kronecker's theorem ([3], Theorem 444), it is possible to find $x = x(y)$ such that

$$\begin{gathered} \sin(\tfrac{1}{2}k\alpha_{j',1}x) = o(\exp[-\tfrac{1}{2}s_{j'}\alpha_{j',1}y]) \\ (y \to \infty) \qquad j' \neq j, k = 1, \cdots, s_{j'} \end{gathered} \tag{2.6}$$

and

$$\sin(\tfrac{1}{2}s_j\alpha_{j,1}x) \geqq 1 - \varepsilon . \tag{2.7}$$

We have then from (2.4)

$$u_1(0, y) - u_1(x, y) = O[l'_{j,s_j} \exp(s_j\alpha_{j,1}y)]$$

when $y \to \infty$ and (2.2) implies (2.5).

Let now $k < s_j$ and let ν be the smallest integer greater than k such that $l'_{j,\nu} > 0$ (if such a ν does not exist, the preceding proof is still valid). From the hypothesis of induction, we can suppose that $l'_{j,k'}$ is zero if k' ($>k$) is not a multiple of ν. From Kronecker's theorem, it is possible to find $x = x(y)$ and an integer p_j such that (2.6) and

$$x\alpha_{j,1} - 2p_j\pi - \frac{2\pi}{\nu} = o(\exp[-\tfrac{1}{2}s_j\alpha_{j,1}y]) \qquad (y \to \infty) \tag{2.8}$$

are satisfied. We have then from (2.4)

$$u_1(0, y) - u_1(x, y) = O[l'_{j,k} \sin^2(\tfrac{1}{2}k\alpha_{j,1}x) \exp(k\alpha_{j,1}y)]$$

and

$$\sin^2(\tfrac{1}{2}k\alpha_{j,1}x) \geqq c > 0 .$$

It follows from (2.2) that

$$l'_{j,k} \geqq 0$$

and the theorem is demonstrated.

We can generalize the Theorem 3 in the following manner:

THEOREM 4. *Let f_0 be the infinitely divisible characteristic function of the variable t defined by*

$$f_0(t) = \exp\left\{i\gamma t + \sum_{j=1}^{p}\sum_{k=1}^{r_j} \lambda_{j,k}[\exp(i\alpha_{j,k}t) - 1] + \sum_{q=1}^{\infty} \mu_q[\exp(i\beta_q t) - 1]\right\} ,$$

where the following conditions are satisfied

(1) γ *is a real constant;*

(2) *the* $\lambda_{j,k}$ *and the* μ_q *are nonnegative constants and there exists a positive constant* K *such that*

$$\mu_q = O[\exp(-K\beta_q^2)] \qquad (q \to \infty) .$$

(3) *the* $\alpha_{j,k}$ *and the* β_q *are positive constants such that*

(a) $\alpha_{j,k+1}/\alpha_{j,k}$ $(k = 1, \cdots, r_j - 1; j = 1, \cdots, p)$ *and* β_{q+1}/β_q $(q = 1, 2, \cdots)$ *are integers greater than* 1;

(b) $\alpha_{1,1}, \cdots, \alpha_{p,1}$ *and* β_1 *are rationally independent. If* f_1 *is a factor of* f_0, *then*

$$f_1(t) = \exp\left\{ict + \sum_{j=1}^{p}\sum_{k=1}^{r_j} l_{j,k}[\exp(i\alpha_{j,k}t) - 1] + \sum_{q=1}^{\infty} m_q[\exp(i\beta_q t) - 1]\right\} ,$$

where c *is a real constant and the* $l_{j,k}$ *and the* m_q *satisfy*

$$0 \leqq l_{j,k} \leqq \lambda_{j,k} ; \quad 0 \leqq m_q \leqq \mu_q .$$

Proof. The proof is essentially the same as the preceding. Using the Theorem 2 of the introduction, we obtain the representation

$$f_1(t) = \exp\Big\{ict + \sum_{j=1}^{p}\sum_{k=1}^{s_j} l'_{j,k}[\exp(ik\alpha_{j,1}t) - 1] + \sum_{q=1}^{\sigma} m'_q[\exp(iq\beta_1 t) - 1] + \sum_{q=\tau}^{\infty} m_q[\exp(i\beta_q t) - 1]\Big\} ,$$

where c, the $l'_{j,k}$ and the m'_q are real constants, the m_q satisfy

$$0 \leqq m_q \leqq \mu_q$$

and where s_j, σ and τ are defined by $(d = \sup_j \alpha_{j,r_j})$

$$s_j\alpha_{j,1} \leqq d < (s_j + 1)\alpha_{j,1} ,$$
$$\sigma\beta_1 \leqq d < (\sigma + 1)\beta_1 ,$$
$$\beta_{\tau-1} \leqq d < \beta_\tau .$$

The proof of the nonnegativity of all the $l'_{j,k}$ and of all the $m'_q(q \leqq \sigma)$ (which implies that all the $l'_{j,k}$ for $k\alpha_{j,1} \notin \{\alpha_{j,k}\}$ and all the m'_q for $q\beta_1 \notin \{\beta_q\}$ are zero) is the same except that we use instead of the Theorem 444 of [3]) the other form of Kronecker's theorem (Theorem 443 of [3]) which asserts that the values of x satisfying (2.6) and (2.7) (or (2.6) and (2.8)) can be taken in the form $2\kappa\pi/\beta_q$ (κ integer).

3. Some auxiliary results. We enumerate now some results which are useful in the following section.

LEMMA 1. ([7], Corollary of the Theorem 1). *Let f be a function of the complex variable z, analytic in the half-plane* $\operatorname{Re} z \geqq 0$ *and satisfying the conditions*

(1) $|f(z)| \leqq M_1 |z+1|^a$ *for* $\operatorname{Re} z = 0$,

(2) $|f(z)| \leqq M_2(z+1)^c \exp(bz)$ *for* $\operatorname{Im} z = 0$,

(3) $|f(z)| \leqq M_3 |z+1|^c \exp[d(\operatorname{Re} z)^2]$ *for* $\operatorname{Re} z \geqq 0$,

where M_1, M_2, M_3 are positive constants and a, b, c ($\geqq a$) and d are nonnegative constants. Then in all the half-plane $\operatorname{Re} z \geqq 0$

$$|f(z)| \leqq M_1 |z+1|^a \exp(b \operatorname{Re} z) .$$

LEMMA 2. *Let f be a function of the n complex variables $z = (z_1, \cdots, z_n)$ admitting the representation*

$$f(z) = \sum_{p_1=0}^{\infty} \cdots \sum_{p_n=0}^{\infty} d_{p_1,\dots,p_n} \exp\left(2\pi \sum_{j=1}^{n} \frac{p_j z_j}{T_j}\right) ,$$

where $T_j > 0$ ($j = 1, \cdots, n$). In order that the constants $d_{p_1,\dots,p_n}$ satisfy for some $K > 0$ the relation

$$d_{p_1,\dots,p_n} = O\left(\exp\left(-K \sum_{j=1}^{n} p_j^2\right)\right) \qquad \left(\sum_{j=1}^{n} p_j \to \infty\right) ,$$

it is necessary that f be an entire function satisfying

$$\ln |f(z)| = O\left(\sum_{j=1}^{n} |\operatorname{Re} z_j|^2\right) \qquad (|z| \to \infty)$$

and sufficient that f be an entire function satisfying

$$\ln |f(z)| = O\left(\sum_{j=1}^{n} |\operatorname{Re} z_j|^2 + \ln |z|\right)$$

$$(|z| \to \infty) \qquad \left(|z|^2 = \sum_{j=1}^{n} |z_j|^2\right) .$$

In the case $n = 1$, this lemma is a particular case of the Theorem 2 of [7]. The proof in the general case is the same as in [7] and is therefore omitted.

LEMMA 3. *If the entire function f of the variable z satisfies for some real K the condition*

$$|f(z)| \leqq \exp[K \operatorname{Re} z + O(\ln |z|)] \qquad (\operatorname{Re} z \geqq 0)$$

when $|z| \to \infty$ and admits an expansion of the form

$$f(z) = \sum_{p=-\infty}^{+\infty} a_p \exp(2\pi p z / T) ,$$

where $T > 0$ *and the series converges uniformly in every bounded set, then*

$$a_p = 0$$

for

$$p > [KT(2\pi)^{-1}] .$$

This lemma is a particular case of the Theorem 3 of [7].

LEMMA 4. *If* φ *is an entire function of the* n *variables* $z = (z_1, \cdots, z_n)$ *such that for any* $x, y \in R^n$ *and any* $\varepsilon > 0$

$$u(x, y) = \operatorname{Re} \varphi(x + iy) = O[\exp{(\tau + \varepsilon)(|x| + |y|)}] \quad (|x| + |y| \to \infty) ,$$

then φ *is a function of exponential type* τ *with respect to the hermitian norm.*

This lemma is a particular case of the Lemma 2 of the theorem 2.5 of [1].

For the following lemma, we recall the

DEFINITION. Let $\{f_n\}$ be a sequence of functions belonging to a Banach space of functions. The f_n are said topologically independent if the relation

$$\lim_{\varepsilon \to 0} \left\| \sum_{n=1}^{\infty} \alpha_n(\varepsilon) f_n \right\| = 0$$

implies

$$\lim_{\varepsilon \to 0} \alpha_n(\varepsilon) = 0 \qquad n = 1, 2, \cdots .$$

We have then the

LEMMA 5. (Lemma 1 of the Theorem 6.1 of [1]). *Let* $\{\lambda_j\}$ *a sequence of real numbers such that*

$$\sum_{j=1}^{\infty} \frac{1}{|\lambda_j|} < +\infty .$$

Then the functions $1, z, \exp(\lambda_j z)$ $(j = 1, 2, \cdots)$ *are topologically independent in the space* $C(a, b)$ *of continuous functions* f *on* $[a, b]$ $(-\infty < a < b < +\infty)$ *with the norm* $\|f\| = \sup_{a<z<b} |f(z)|$.

Recall ([1], Chapter 4) that a function φ of the n complex variables $z = (z_1, \cdots, z_n)$ is said a ridge function if it is an entire function satisfying for any $z \in C^n$ the relation

$$|\varphi(z)| \leqq \varphi(\operatorname{Re} z) \qquad (\operatorname{Re} z = (\operatorname{Re} z_1, \cdots, \operatorname{Re} z_n)) .$$

We have the

LEMMA 6. *Let φ_0 be a ridge function of the n variables $z = (z_1, \cdots, z_n)$ without zeros and φ_1 and φ_2 be two ridge functions such that*

$$\varphi_0 = \varphi_1 \varphi_2 .$$

There exists a positive constant C such that

$$M(r; \log \varphi_j) \leqq 6rM(r+1; \log \varphi_0) + Cr(r+1) \qquad (j = 1, 2) ,$$

where

$$M(r; f) = \sup_{|z|=r} |f(z)| .$$

Proof. Let

$$\psi_j(z) = \log [\varphi_j(z)] , \qquad \operatorname{Re} \psi_j(x+iy) = u_j(x, y)$$

for any $x, y \in R^n$ ($j = 0, 1, 2$). Since φ_1 and φ_0/φ_1 are ridge functions without zeros, we have

$$0 \leqq u_1(x, 0) - u_1(x, y) \leqq u_0(x, 0) - u_0(x, y) \leqq 2M(r; \psi_0) \tag{3.1}$$

for $|x + iy| \leqq r$.

We estimate now $|u_1(x, 0)|$. For that, we use the existence for any ridge function φ of a positive constant C_φ such that

$$\log \varphi(x) \geqq -C_\varphi |x| \tag{3.2}$$

for any $x \in R^n$. Indeed, since $\log \varphi(\lambda\theta)$ is for any direction θ of R^n a convex function of λ, we have

$$\log \varphi(\lambda\theta) \geqq \log \varphi(0) + \lambda(\alpha \cdot \theta) , \tag{3.3}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_j \varphi(0) = \{\partial\varphi(0)/\partial z_j$ and where $(\alpha \cdot \theta)$ indicates the scalar product of the vectors α and θ. The relation (3.2) is an immediate consequence of (3.3).

From (3.2), we have

$$u_1(x, 0) = \psi_1(x) \geqq -C_1 |x| , \tag{3.4}$$

$$u_1(x, 0) = \psi_0(x) - \psi_2(x) \leqq \psi_0(x) + C_2 |x| . \tag{3.5}$$

From (3.4) and (3.5), it follows

$$|u_1(x, 0)| \leqq M(r; \psi_0) + Cr \qquad (|x| \leqq r)$$

for some positive constant C and from (3.1)

$$|u_1(x, y)| \leqq 3M(r; \psi_0) + Cr \, (|x + iy| \leqq r) .$$

Let now g_θ the function of the complex variable λ defined by

$$g_\theta(\lambda) = \psi_1(\lambda\theta)$$

for some direction θ of R^n. Then

$$v_\theta(\mu, \nu) = \operatorname{Re} g_\theta(\mu + i\nu) = u_1(\mu\theta + i\nu\theta)$$

for any real μ and ν. We have then $(\lambda = \mu + i\nu)$

$$g'_\theta(\lambda) = \frac{1}{\pi}\int_0^{2\pi} v_\theta(\mu + \cos\alpha, \nu + \sin\alpha)e^{-i\alpha}d\alpha\ ;$$

so that

$$(3.6)\qquad \begin{aligned} |g'_\theta(\lambda)| &\leqq 2 \sup_{0\leqq\alpha\leqq 2\pi} |v_\theta(\mu + \cos\alpha, \nu + \sin\alpha)| \\ &\leqq 6M(r+1; \log\varphi_0) + C(r+1) \end{aligned}$$

$(|\lambda| \leqq r)$ for some positive constant C. Since

$$g_\theta(\lambda) = \int_0^\mu g'_\theta(\xi)d\xi + i\int_0^\nu g'_\theta(\mu + i\nu)d\eta$$

and since θ is arbitrary, the lemma is a consequence of (3.6).

LEMMA 7. (Lemma of the Theorem 5 of [2]). *If f is an entire characteristic function of the two variables t_1 and t_2 and t_2^0 a real constant, the function $f_{t_2^0}$ defined by*

$$f_{t_2^0}(t_1) = \frac{f(t_1, it_2^0)}{f(0, it_2^0)}$$

is an entire characteristic function.

4. The case of several variables. First of all, we consider the case of functions of two variables.

THEOREM 5. *Let f_0 be the infinitely divisible characteristic function of the two variables $t = (t_1, t_2)$ defined by*

$$\begin{aligned} f_0(t) = \exp\Big\{i\pi(t) + \sum_{j=1}^{\infty}(\lambda_j[\exp(i\alpha_j t_1) - 1] \\ + \mu_j[\exp(i\beta_j t_2) - 1] + \nu_j[\exp(i\alpha_j t_1 + i\beta_j t_2) - 1])\Big\}, \end{aligned}$$

where the following conditions are satisfied

(1) *π is an homogeneous polynomial of degree one with real coefficients;*

(2) *λ_j, μ_j, ν_j are nonnegative constants and there exists a positive constant K such that*

$$\lambda_j = O[\exp(-K\alpha_j^2)] ; \qquad \mu_j = O[\exp(-K\beta_j^2)] ,$$
$$\nu_j = O[\exp(-K(\alpha_j^2 + \beta_j^2))] \qquad (j \to +\infty) ;$$

(3) *the* α_j *are positive constants satisfying the three conditions*

(a) *there exists* q_1 *such that* α_{j+1}/α_j *is an integer greater than* 1 *for* $j \geqq q_1$;

(b) *the set* $\{\alpha_j; j < q_1\}$ *can be decomposed in* p *sets* $\{\alpha_{j,k}\}$ $(j = 1, \cdots, p;\ k = 1, \cdots, r_j;\ \sum_{j=1}^{p} r_j = q_1 - 1)$ *such that* $\alpha_{j,k+1}/\alpha_{j,k}$ $(k = 1, \cdots, r_j - 1;\ j = 1, \cdots, p)$ *is an integer greater than* 1 *and* $\alpha_{1,1}, \cdots, \alpha_{p,1}$ *are rationally independent*;

(c) *either* α_{q_1} *is a multiple of one of the* α_{j,r_j} *or* $\alpha_{1,1}, \cdots, \alpha_{p,1}$ *and* α_{q_1} *are rationally independent*;

(4) *the* β_j *are positive constants having the same property.*

If f_1 *is a factor of* f_0, *then*

$$f_1(t) = \exp\Big\{iP(t) + \sum_{j=1}^{\infty} (l_j[\exp(i\alpha_j t_1) - 1]$$
$$+ m_j[\exp(i\beta_j t_2) - 1] + n_j[\exp(i\alpha_j t_1 + i\beta_j t_2) - 1])\Big\}$$

where P *is an homogeneous polynomial of degree one with real coefficients and where* l_j, m_j, n_j *are constants satisfying the conditions*

$$0 \leqq l_j \leqq \lambda_j ; \qquad 0 \leqq m_j \leqq \mu_j , \qquad 0 \leqq n_j \leqq \nu_j .$$

Proof. Let f_1 and f_2 be the two characteristic functions such that for any real t_1 and t_2

$$f_0(t_1, t_2) = f_1(t_1, t_2) f_2(t_1, t_2) . \tag{4.1}$$

Since f_0 is an entire characteristic function, from Raikov's theorem ([1], Theorem 2.3), f_j is also entire $(j = 1, 2)$ and the equation (4.1) is also valid for any complex t_1 and t_2, Letting

$$\varphi_j(z) = f_j(-iz) ,$$
$$u_j(x, y) = \operatorname{Re} \log \varphi_j(x + iy) ,$$

$(j = 0, 1, 2)$ for any $x, y \in R^2$, since φ_j is a ridge function ([1], Corollary 1 of the Theorem 2.1), we have

$$0 \leqq u_1(x, 0) - u_1(x, y) \leqq u_0(x, 0) - u_0(x, y) \tag{4.2}$$

for any $x, y \in R^2$.

If we fix z_2 real, using the Lemma 7 and the Theorem 4, we have

$$\log \varphi_1(z) = a + bz_1 + \sum_{j=1}^{\infty} c_j \exp(\alpha_j z_1) , \tag{4.3}$$

where a, b, c_j are functions of z_2, real for z_2 real and satisfying

$$(4.4) \qquad 0 \leqq c_j(z_2) \leqq \lambda_j + \nu_j \exp(\beta_j z_2) .$$

If we fix z_1 real, we have

$$(4.5) \qquad \log \varphi_1(z) = r + s z_2 + \sum_{j=1}^{\infty} t_j \exp(\beta_j z_2) ,$$

where r, s, t_j are functions of z_1, real for z_1 real and satisfying

$$(4.6) \qquad 0 \leqq t_j(z_1) \leqq \mu_j + \nu_j \exp(\alpha_j z_1) .$$

From (4.3) and (4.5), we obtain the equation for any real z_1 and z_2

$$a + b z_1 + \sum_{j=1}^{\infty} c_j \exp(\alpha_j z_1) = r + s z_2 + \sum_{k=1}^{\infty} t_2 \exp(\beta_k z_2)$$

which can be solved by using the Lemma 5 (for the details, see the proof of the Theorem 6.1 of [1]). We obtain for any z_1 and z_2 complex the representation

$$(4.7) \qquad \begin{aligned} \log \varphi_1(z) = c + P(z) + d z_1 z_2 + \sum_{j=1}^{\infty} [\rho_j z_2 \exp(\alpha_j z_1) + \sigma_j z_1 \exp(\beta_j z_2)] \\ + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} n_{j,k} \exp(\alpha_j z_1 + \beta_k z_2) , \end{aligned}$$

where all the constants and the coefficients of the homogeneous polynomial P of degree one are real (with the convention $\alpha_0 = \beta_0 = n_{0,0} = 0$). By an elementary computation, we obtain

$$(4.8) \qquad \begin{aligned} u_1(x, 0) - u_1(x, y) = d y_1 y_2 + \sum_{j=1}^{\infty} [2\rho_j x_2 \exp(\alpha_j x_1) \sin^2(\tfrac{1}{2}\alpha_j y_1) \\ + 2\sigma_j x_1 \exp(\beta_j x_2) \sin^2(\tfrac{1}{2}\beta_j y_2) \\ + \rho_j y_2 \exp(\alpha_j x_1) \sin(\alpha_j y_1) \\ + \sigma_j y_1 \exp(\beta_j x_2) \sin(\beta_j y_2)] \\ + 2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} n_{j,k} \exp(\alpha_j x_1 + \beta_k x_2) \sin^2\left(\frac{\alpha_j y_1 + \beta_k y_2}{2}\right) . \end{aligned}$$

Letting $|y_1| \to \infty$, we obtain from (4.2) and (4.8)

$$d y_2 + \sum_{j=1}^{\infty} \sigma_j \exp(\beta_j x^2) \sin(\beta_j y_2) = 0 .$$

Since the expression in the left member is the imaginary part of

$$d z_2 + \sum_{j=1}^{\infty} \sigma_j \exp(\beta_j z_2) ,$$

we deduce from the Lemmas 4 and 3 that

$$d = \sigma_j = 0 .$$

In the same manner, letting $|y_2| \to \infty$, we obtain

$$\rho_j = 0 .$$

From the Lemma 5 and (4.4), it follows that for any real x_2

$$0 \leqq \sum_{k=0}^{\infty} n_{j,k} \exp(\beta_k x_2) \leqq \lambda_j + \nu_j \exp(\beta_j x_2) . \tag{4.9}$$

On the other hand, $\log \varphi_0$ satisfies

$$\log \varphi_0(z) = O[|z|(1 + \exp(N|\operatorname{Re} z|^2))] \qquad (|z| \to \infty)$$

for some $N > 0$. It follows from the lemma 6 that

$$\log \varphi_1(z) = O[|z|^2(1 + \exp(N|\operatorname{Re} z|^2))] \qquad (|z| \to \infty)$$

and from the sufficient part of the Lemma 2 applied to

$$\sum_{j=q_1}^{\infty} \sum_{k=q_2}^{\infty} n_{j,k} \exp(\alpha_j z_1 + \beta_k z_2)$$

(the constant q_1 is defined in the statement of the theorem and the constant q_2 is the analogous for the β_k), we have for some $\kappa' > 0$

$$n_{j,k} = O[\exp(-\kappa'(j^2 + k^2))] \qquad (|j| + |k| \to \infty) ,$$

that implies from the necessary part of the Lemma 2

$$\sum_{k=0}^{\infty} n_{j,k} \exp(\beta_k z_2) = O[\exp(N'|\operatorname{Re} z_2|^2)] \qquad (|z_2| \to \infty)$$

for any complex z_2 and some $N' > 0$. Using the Lemma 1, we obtain

$$\sum_{k=0}^{\infty} n_{j,k} \exp(\beta_k z_2) = O[\exp(\beta_j z_2)] \qquad (|z_2| \to \infty)$$

in $\{\operatorname{Re} z_2 \geqq 0\}$, that implies from the Lemma 3

$$n_{j,k} = 0$$

for all the k such that $\beta_k > \beta_j$ and from (4.9)

$$n_{j,j} \geqq 0 , \qquad n_{j,0} \geqq 0 .$$

In the same manner, from (4.6), we obtain

$$n_{j,k} = 0$$

for all the j such that $\alpha_j > \alpha_k$ and

$$n_{0,j} \geqq 0 .$$

In particular, we have ($q = \sup(q_1, q_2)$)

$$n_{j,k} = 0$$

if $(j, k) \notin \{(j, j), (0, j), (j, 0)\}$ and either $j \geqq q$ or $k \geqq q$.

(4.7) becomes (with $l_j = n_{j,0}, m_j = n_{0,j}, n_j = n_{j,j}$ if $j \geqq q$).

$$\begin{aligned} \log \varphi_1(z) &= c + P(z) \\ &+ \sum_{j=q}^{\infty} [l_j \exp(\alpha_j z_1) + m_j \exp(\beta_j z_2) + n_j \exp(\alpha_j z_1 + \beta_j z_2)] \quad (4.10) \\ &+ \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} n_{j,k} \exp(\alpha_j z_1 + \beta_k z_2) \end{aligned}$$

and (4.8) becomes

$$\begin{aligned} &\tfrac{1}{2}[u_1(x, 0) - u_1(x, y)] \\ &= \sum_{j=q}^{\infty} \Big[l_j \exp(\alpha_j x_1) \sin^2(\tfrac{1}{2}\alpha_j y_1) + m_j \exp(\beta_j x_2) \sin^2(\tfrac{1}{2}\beta_j y_2) \\ &\quad + n_j \exp(\alpha_j x_1 + \beta_j x_2) \sin^2\Big(\frac{\alpha_j y_1 + \beta_j y_2}{2}\Big)\Big] \quad (4.11) \\ &\quad + \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} n_{j,k} \exp(\alpha_j x_1 + \beta_k x_2) \sin^2\Big(\frac{\alpha_j y_1 + \beta_k y_2}{2}\Big) . \end{aligned}$$

We show now by induction that all the $n_{j,k}$ $(i \leqq q - 1, k \leqq q - 1)$ are nonnegative (that implies $n_{j,k} = 0$ if $(j, k) \notin \{(j, j), (j, 0), (0, j)\}$). We show that this result is true for $j = j_0$ such that $\alpha_{j_0} = \sup_{j=1\cdots,q-1} \alpha_j$. We put $y_1 = 2\pi/\alpha_q$ and choose y_2 from Kronecker's theorem (Theorem 443 of [3]) such that

(a) $$y_2 = \frac{2\kappa'\pi}{\beta_{l'}}$$

(κ' integer) where $\beta_{l'}$ is the smallest number greater than β_k such that $\beta_q/\beta_{l'}$ is integer;

(b) $$\sin\Big(\frac{\alpha_{j_0} y_1 + \beta_{k'} y_2}{2}\Big) = o[\exp(\tfrac{1}{2}\beta_{k'} x_2)] \qquad (x_2 \longrightarrow \infty)$$

for all the k' such that $\beta_{k'} \geqq \beta_k$;

(c) $$\sin\Big(\frac{\alpha_{j_0} y_1 + \beta_k y_2}{2}\Big) \geqq C > 0 .$$

Then if x_2 is chosen great enough, we obtain from (4.11)

$$u_1(x, 0) - u_1(x, y) = O\Big[n_{j_0,k} \exp(\alpha_{j_0} x_1 + \beta_k x_2) \sin^2\Big(\frac{\alpha_{j_0} y_1 + \beta_k y_2}{2}\Big)\Big]$$

$(x_1 \longrightarrow \infty)$, that implies with (4.2)

$$n_{j_0 k} \geqq 0 .$$

Let now (j, k) arbitrary. We can suppose that

$$n_{j',k'} \geqq 0 \quad \text{if} \quad (j', k') \in \{(j', j'), (j', 0), (0, j')\}$$

and

$$n_{j',k'} = 0 \quad \text{if} \quad (j', k') \notin \{(j', j'), (j', 0), (0, j')\}$$

if either $\alpha_{j'} > \alpha_j$ or $j' = j$, $\beta_{k'} > \beta_k$. Then we choose y_1 from Kronecker's theorem such that

(a) $$y_1 = \frac{2\kappa\pi}{\alpha_l}$$

(κ integer) where α_l is the smallest integer greater than α_j such that α_q/α_l is integer;

(b) $$\sin(\tfrac{1}{2}\alpha_{j'}y_1) = o[\exp(-\tfrac{1}{2}\alpha_{j'}x_1)] \qquad (x_1 \to \infty)$$

for all j' such that $\alpha_{j'} > \alpha_j$;

(c) $$|\sin(\tfrac{1}{2}\alpha_j y_1)| \geqq c > 0 .$$

We choose now y_2 such that, from Kronecker's theorem

(α) $$y_2 = \frac{2\kappa'\pi}{\beta_{l'}}$$

(κ' integer) where $\beta_{l'}$ is the smallest integer greater than β_k such that $\beta_q/\beta_{l'}$ is integer;

(β) $$\sin\left(\frac{\alpha_{j'}y_1 + \beta_{j'}y_2}{2}\right) = o[\exp(-\tfrac{1}{2}\alpha_{j'}x_1)] \qquad (x_1 \to \infty)$$

for all j' such that $\alpha_{j'} > \alpha_j$;

(γ) $$\sin\left(\frac{\alpha_j y_1 + \beta_j y_2}{2}\right) = o[\exp(-\tfrac{1}{2}\alpha_j x_1)] \qquad (x_1 \to \infty)$$

if $\beta_j > \beta_k$ (otherwise, this condition is superfluous);

(δ) $$\left|\sin\left(\frac{\alpha_j y_1 + \beta_k y_2}{2}\right)\right| \geqq C' \geqq 0 .$$

We have then, from (4.11), if x_2 is chosen great enough,

$$u_1(x, 0) - u_1(x, y) = O\left[n_{j,k} \exp(\alpha_j x_1 + \beta_k x_2) \sin^2\left(\frac{\alpha_j y_1 + \beta_k y_2}{2}\right)\right]$$

$(x_1 \to \infty)$, that implies

$$n_{j,k} \geqq 0 ,$$

and the theorem is demonstrated, the value of c in (4.10) being determined by the condition $\log \varphi_1(0) = 0$.

From this theorem, we deduce easily by the method of the Chapters 5 and 6 of [1] the

THEOREM 6. *Let f_0 be the infinitely divisible characteristic function of the n variables $t = (t_1, \cdots, t_n)$ defined by*

$$f_0(t) = \exp\left\{i\pi(t) + \sum_{j=1}^{\infty}\sum_{\varepsilon}\lambda_{j,\varepsilon}\left[\exp\left(i\sum_{k=1}^{n}\varepsilon_k\alpha_{j,k}t_k\right) - 1\right]\right.$$

where the following conditions are satisfied:

(1) *π is an homogeneous polynomial of degree one with real coefficients;*

(2) *$\varepsilon_k = 0$ or 1 and $\sum_{\varepsilon}$ indicates the summation on the $2^n - 1$ values of $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ different from $(0, \cdots, 0)$;*

(3) *$\lambda_{j,\varepsilon}$ are nonnegative constants and there exists a positive constant K such that*

$$\lambda_{j,\varepsilon} = O\left[\exp\left(-K\sum_{k=1}^{n}\varepsilon_k\alpha_{j,k}^2\right)\right] \qquad (j \to +\infty)\ ;$$

(4) *$\{\alpha_{j,k}\}$ is, for $k = 1, \cdots, n$, a sequence of positive numbers satisfying the condition* (3) *of the Theorem* 5.

If f_1 is a factor of f_0, then

$$f_1(t) = \exp\left\{iP(t) + \sum_{j=1}^{\infty}\sum_{\varepsilon}l_{j,\varepsilon}\left[\exp\left(i\sum_{k=1}^{n}\varepsilon_k\alpha_{j,k}t_k\right) - 1\right]\right\},$$

where P is an homogeneous polynomial of degree one with real coefficients and $l_{j,\varepsilon}$ are constants satisfying the conditions

$$0 \leqq l_{j,\varepsilon} \leqq \lambda_{j,\varepsilon}\ .$$

With the same method, we can deduce from the Theorem 1′ of Ostrovskiy [8] the

THEOREM 7. *Let f_0 be the infinitely divisible characteristic function of the n variables $t = (t_1, \cdots, t_n)$ defined by*

$$f_0(t) = \exp\left\{i\pi(t) + \sum_{j=1}^{\infty}\sum_{\varepsilon}\lambda_{j,\varepsilon}\left[\exp\left(i\sum_{k=1}^{n}\varepsilon_k\alpha_{j,k}t_k\right) - 1\right]\right.$$

where, beyond the conditions (1), (2), (3) *of the preceding theorem, the following condition is satisfied:*

(4′) *$\{\alpha_{j,k}\}$ is for $k = 1, \cdots, n$ a sequence of increasing positive numbers such that*

(a) *there exists q_k such that $\alpha_{j+1,k}/\alpha_{j,k}$ $(j \geqq q_k)$ is an integer* *greater than* 1;

(b) *there exists a positive constant* a_k *such that* $a_k \leqq \lambda_{j,k} \leqq 2a_k$ $(j < q_k)$.

If f_1 *is a factor of* f_0, *then*

$$f_1(t) = \exp\left\{iP(t) + \sum_{j=1}^{\infty}\sum_{\varepsilon} l_{j,\varepsilon}\left[\exp\left(i\sum_{k=1}^{n}\varepsilon_k\alpha_{j,k}t_k\right) - 1\right]\right\},$$

where P *is an homogeneous polynomial of degree one with real coefficients and* $l_{j,\varepsilon}$ *are constants satisfying the conditions*

$$0 \leqq l_{j,\varepsilon} \leqq \lambda_{j,\varepsilon}.$$

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