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NILPOTENCY CLASS OF A MAP AND STASHEFF'S CRITERION

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Let $f: X \to Y$ be a map and let $e: \Sigma\Omega X \to X$ be the map whose adjoint is $1_{\Omega X}$. Then we prove the following results.

THEOREM 1. $\operatorname{nil} f \leqq 1$ **if and only if** $fe\nabla: \Sigma\Omega X \vee \Sigma\Omega X \to Y$ **can be extended to** $\Sigma\Omega X \times \Sigma\Omega X$.

THEOREM 2. **Let X be an H'-space. Then** $\operatorname{nil} f \leqq 1$ **if and only if** $f\nabla: X \vee X \to Y$ **can be extended to** $X \times X$.

THEOREM 3. $\operatorname{nil} f = \operatorname{nil}(fe)$.

Theorem 1 may be regarded as an extension of Stasheff's criterion for a loop space to be homotopy-commutative. These theorems may all be regarded as extensions of Stasheff's criterion in various ways. We also discuss the duals of these results. Theorem 3 dualises, but the others do not. A sample result in the dual situation is

THEOREM. $\operatorname{conil} f \leqq \Sigma w \operatorname{cat}(e'f)$ **where** $e': Y \to \Omega\Sigma Y$ **is the adjoint of** $1_{\Sigma Y}$.

In this paper we shall work in the category $\mathscr{T}$ of spaces with base point and having the homotopy type of countable CW complexes. All maps and homotopies shall respect base points. The maps of our category $\mathscr{T}$ shall be homotopy classes of maps, but for simplicity we shall use the same symbol for a map and its homotopy class. Given spaces X, Y, we denote the set of homotopy classes of maps from X to Y by $[X, Y]$. We have an isomorphism $\tau: [\Sigma X, Y] \to [X, \Omega Y]$ where Σ, Ω are the suspension and loop functors respectively. We denote $\tau(1_{\Sigma X})$ by e' and $\tau^{-1}(1_{\Omega X})$ by e.

1. For convenience let us recall some notions of Peterson's theory of structures [7]. We shall follow the definitions and notations of [4]. Let $\mathscr{C}$ be a category. By a left structure system $\mathscr{L}$ over $\mathscr{C}$ we mean $\mathscr{L} = (L, W, S; d, j)$ where $L, W, S: \mathscr{C} \to \mathscr{T}$ are covariant functors and $d: W \to L$, $j: W \to S$ are natural transformations. Given an object X of $\mathscr{C}$ we say that X is $\mathscr{L}$-structured if there exists a map $\varphi: SX \to LX$ such that $\varphi j(X) \simeq d(X)$. Given a category $\mathscr{C}$, we have a category $\mathscr{C}^2$ of pairs. An object of $\mathscr{C}^2$ is a map $f: X \to Y$ of $\mathscr{C}$, and given objects $f: X_1 \to X_2$, $g: Y_1 \to Y_2$ of $\mathscr{C}^2$, a map (u, v): $f \to g$ is a pair of maps $u: X_1 \to Y_1$, $v: X_2 \to Y_2$ such that $gu = vf$. We have convariant functors $D_0, D_1: \mathscr{C}^2 \to \mathscr{C}$ given by $D_0(f) = Y$, $D_1(f) =$

X where $f: X \to Y$. Also given $(u, v): f \to g$, we have $D_0(u, v) = v$, $D_1(u, v) = u$. We have a natural trasformation $G: D_1 \to D_0$ given by $G(f) = f$ for $f \in \mathscr{C}^2$. Given a left structure $\mathscr{L} = (L, W, S; d, j)$ over $\mathscr{C}$, we have a left structure $\mathscr{L}^2 = (LD_0, WD_1, SD_1; (dD_0)(WG), jD_1)$ over $\mathscr{C}^2$. Given an object f of $\mathscr{C}^2$, we shall say that f is $\mathscr{L}$-structured if it is $\mathscr{L}^2$-structured. It is easily seen that if $f: X \to Y$ is an object of $\mathscr{C}^2$, and X or Y is $\mathscr{L}$-structured, then f is $\mathscr{L}$-structured.

We have the left structure $H = (1, \bigvee_{i=1}^2, \prod_{i=1}^2, \nabla, j)$ over $\mathscr{T}$, where 1 is the identity functor of $\mathscr{T}$, $\bigvee_{i=1}^2$ is the wedge product, $\prod_{i=1}^2$ is the cartesian product and ∇, j are the folding and inclusion natural transformations respectively. We observe that a space X is H-structured precisely if it is an H-space. Also a map $f: X \to Y$ is H-structured if and only if $f\nabla: X \vee X \to Y$ extends to $X \times X$.

2. Let $\mathscr{L} = (L, W, S; d, j)$ be a left structure system over a category $\mathscr{C}$. Let $f: X \to Y$, $g: Y \to Z$ be maps. Then it is easily seen that if f is $\mathscr{L}$-structured or g is $\mathscr{L}$-structured, then gf is $\mathscr{L}$-structured.

We recall that in [1], there is defined a generalized Whitehead product $[,]: [\Sigma A, X] \times [\Sigma B, X] \to [\Sigma(A \wedge B), X]$ where A, B, X are spaces and $A \wedge B$ is the smashed product. Now suppose X is an H-space. Then we have a generalized Samelson product (see [2]) $\langle , \rangle$: $[A, X] \times [B, X] \to [A \wedge B, X]$. These homotopy operations are related in the following way. Suppose α is an element of $[\Sigma A, X]$, β is an element of $[\Sigma B, X]$ where A, B, X are spaces. Then

$$\tau[\alpha, \beta] = \langle \tau(\alpha), \tau(\beta) \rangle .$$

We shall also make the following convention. Let $f: X \to Y$ be a map. Then we have an H-map $\Omega f: \Omega X \to \Omega Y$. We shall write $\operatorname{nil} f$ for $\operatorname{nil} \Omega f$ (see [3] for definitions). Similarly, we have an H'-map $\Sigma f: \Sigma X \to \Sigma Y$. We shall write $\operatorname{conil} f$ for $\operatorname{conil} \Sigma f$.

THEOREM 1. *Let $f: X \to Y$ be a map. Then* $\operatorname{nil} f \leqq 1$ *if and only if $fe\nabla: \Sigma\Omega X \vee \Sigma\Omega X \to Y$ can be extended to $\Sigma\Omega X \times \Sigma\Omega X$.*

Proof. Let $c: \Omega X \times \Omega X \to \Omega X$ be the basic commutator of ΩX. Then $\operatorname{nil} f \leqq 1$ if and only if $(\Omega f)\, c \simeq *$. Let $i_1, i_2: \Sigma\Omega X \to \Sigma\Omega X \vee \Sigma\Omega X$ be the inclusions in the first and second coordinates respectively. Then we have a generalized Whitehead product

$$[i_1, i_2] \in [\Sigma(\Omega X \wedge \Omega X), \Sigma\Omega X \vee \Sigma\Omega X] .$$

Now $\Sigma\Omega X \times \Sigma\Omega X$ is homotopically equivalent to

$$(\Sigma\Omega X \vee \Sigma\Omega X) \bigcup_{[i_1, i_2]} C\Sigma(\Omega X \times \Omega X)$$

(see [1]), so that $fe\nabla$ extends to $\Sigma\Omega X \times \Sigma\Omega X$ if and only if $fe\nabla[i_1, i_2] = 0$, that is, $[fe, fe] = 0$. Now $\tau[fe, fe] = \langle \Omega f, \Omega f \rangle$ and

$$q^{\#}\langle \Omega f, \Omega f \rangle = c(\Omega f \times \Omega f) \simeq (\Omega f)c$$

where the first c denotes the commutator $\Omega Y \times \Omega X \rightarrow \Omega Y$ and the second c denotes the commutor $\Omega X \times \Omega X \rightarrow \Omega X$ and $q \colon \Omega Y \times \Omega Y \, \Omega Y \wedge \Omega Y$ is the projection. Since τ is an isomorphism and $q^{\#}$ is a monomorphism, it follows that $fe\nabla$ extends to $\Sigma\Omega X \times \Sigma\Omega X$ if and only if $\operatorname{nil} f \leqq 1$.

REMARK. If we take f to be the identity map of X, then the theorem says that $\operatorname{nil} X \leqq 1$ if and only if $e\nabla \colon \Sigma\Omega X \vee \Sigma\Omega X \rightarrow X$ extends to $\Sigma\Omega X \times \Sigma\Omega X$, which is just Stasheff's criterion for the homotopy-commutativity of a loop space (see [8]). We also observe that the statement that $fe\nabla$ extends to $\Sigma\Omega X \times \Sigma\Omega X$ is just the statement that fe can be H-structured.

THEOREM 2. *Let $f \colon X \rightarrow Y$ be a map where X is an H'-space. Then* $\operatorname{nil} f \leqq 1$ *if and only if $f\nabla \colon X \vee X \rightarrow Y$ can be extended to $X \times X$.*

In view of the fact that $f\nabla$ can be extended if and only if f can be H structured, Theorem 2 will follow from Theorem 1 and the following lemma.

LEMMA. *Let $f \colon X \rightarrow Y$ be a map where X is an H'-space. Then f is H-structured if and only if $fe \colon \Sigma\Omega X \rightarrow Y$ is H-structured.*

Proof. We need only show that if fe is H-structured then f is H-structured. Suppose fe can be H-structured. Then we can find a map $\varphi \colon \Sigma\Omega X \times \Sigma\Omega X \rightarrow Y$ such that $\varphi j \simeq \nabla(fe \vee fe) = fe\nabla$. Since X is an H':space we have a map $s \colon X \rightarrow \Sigma\Omega X$ such that $es \simeq 1_X$. Then $\varphi(s \times s) \colon X \times X \rightarrow Y$ is an H-structure for f. In fact $\varphi(s \times s)j = \varphi j(s \vee s) \simeq fe\nabla(s \vee s) = fes\nabla \simeq f\nabla$.

REMARK. Theorems 1 and 2 imply that $\operatorname{nil} e \leqq 1$ if and only if ΩX is homotopy-commutative, that is, if and only if $\operatorname{nil} X \leqq 1$. In fact, we always have $\operatorname{nil} X = \operatorname{nil} e$. This fact follows from the next result.

THEOREM 3. *Let $f \colon X \rightarrow Y$ be a map. Then* $\operatorname{nil} f = \operatorname{nil}(fe)$.

Proof. Since we always have $\operatorname{nil}(fe) \leqq \operatorname{nil} f$, it suffices to show that $\operatorname{nil} f \leqq \operatorname{nil}(fe)$. Suppose $\operatorname{nil}(fe) \leqq n$. Then $(\Omega f)(\Omega e)c_{n+1} \simeq *$

where $c_{n+1}\colon (\Omega\Sigma\Omega X)^{n+1} \to \Omega\Sigma\Omega X$ is the commutator map of weight $(n+1)$. Then we have

$$(\Omega f)c_{n+1}(\Omega e \times \cdots \times \Omega e) \simeq *$$

where $c_{n+1}\colon (\Omega X)^{n+1} \to \Omega X$ is also the commutator map of weight $(n+1)$. Consider the map $e'\colon \Omega X \to \Omega\Sigma\Omega X$ such that $e' = \tau(1_{\Omega\Sigma X})$. Clearly $(\Omega e)e' = 1_{\Omega\Sigma}$. Hence we have $(\Omega f)c_{n+1} \simeq *$, that is, $\operatorname{nil} f \leqq n$. This proves the theorem.

3. We now consider the dual situation. It is clear that Theorem 3 dualises immediately to give the following result.

THEOREM 4. *Let $f\colon X \to Y$ be a map and let $e'\colon Y \to \Omega\Sigma Y$ be the adjoint of $1_{\Sigma X}$. Then* $\operatorname{conil} f = \operatorname{conil}(e'f)$.

Let us first define a right structure system over a category $\mathscr{C}$. By this we shall mean $\mathscr{R} = (R, P, T; d, j)$ where $R, P, T\colon \mathscr{C} \to \mathscr{T}$ are covariant functors and $d\colon R \to P$, $j\colon T \to P$ are natural transformations. Given an object $X \in \mathscr{C}$, we say that X is $\mathscr{R}$-structured if there exists a map $\varphi\colon RX \to TX$ such that $j(X)\varphi \simeq d(X)$. Given a right structure $\mathscr{R} = (R, P, T; d, j)$ over $\mathscr{C}$, we can form a right structure $\mathscr{R}^2 = (RD_1, PD_0, TD_0; (dD_0)(RG), jD_0)$ over $\mathscr{C}^2$. We shall say that an element $f\colon X \to Y$ of $\mathscr{C}^2$ is $\mathscr{R}$-structured if it is $\mathscr{R}^2$-structured. It is easily checked that if X or Y is $\mathscr{R}$-structured, then f is $\mathscr{R}$-structured.

The dual of the H-structure is the H'-structure $(1, \prod_{i=1}^{2}, \bigvee_{i=1}^{2}; \Delta, j)$, a right structure over $\mathscr{T}$. Clearly a space X is H'-structured if and only if it is an H'-space. Also a map $f\colon X \to Y$ is H'-structured if and only if $\Delta f\colon X \to Y^2$ can be compressed into $Y \vee Y$. The dual of Theorem 1 would read: $\operatorname{conil} f \leqq 1$ if and only if $\Delta e'f\colon X \to (\Omega\Sigma Y)^2$ can be compressed into $\Omega\Sigma Y \vee \Omega\Sigma Y$. This, however, is false (see [5]). But in this case, we can generalize the H'-structure to another familiar right structure, namely the n-cat structure $(1, \prod_{i=1}^{n+1}; T_1, \Delta, j)$ over $\mathscr{T}$, where T_1 is the fat wedge functor. Thus the 1-cat structure is precisely the H'-structure. Given a space X, we have $\operatorname{cat} X \leqq n$ if there exists a map $\varphi\colon X \to T_1(X, \cdots, X)$ such that $j\varphi \simeq \Delta\colon X \to X^{n+1}$. Given a map $f\colon X \to Y$, we have $\operatorname{cat} f \leqq n$ if $\Delta f\colon X \to Y^{n+1}$ can be compressed into $T_1(Y, \cdots, Y)$.

Given a right structure system $\mathscr{R} = (R, P, T; d, j)$ over $\mathscr{C}$, let us consider the cofibre of $j\colon T \to P$. Suppose the cofibre of j is $q\colon P \to Q$. Let $j_w \to P$ be the fibre of q. Then we obtain a right structure system $\mathscr{R}_w = (R, P, T_w; d, j_w)$ over $\mathscr{C}$, called the associated weak structure. We shall say that an object $X \in \mathscr{C}$ is weakly $\mathscr{R}$-

structured if it can be $\mathscr{R}_w$-structured. Clearly, given a map $f: X \to Y$ we have $w \operatorname{cat} f \leqq n$ if $q\Delta f \simeq *$ where $q: Y^{n+1} \to \bigwedge_{i=1}^{n+1} Y$ is the projection onto the smashed product. Given a right structure $\mathscr{R} = (R, P, T; d, j)$ over $\mathscr{C}$, we have a right structure $\Sigma\mathscr{R} = (\Sigma R, \Sigma P, \Sigma T; \Sigma d, \Sigma j)$ over $\mathscr{C}$, where Σ is the suspension functor. Clearly, if f is $\mathscr{R}$-structured, it is $\Sigma\mathscr{R}$-structured and it is weakly $\mathscr{R}$-structured. Thus $\Sigma w \operatorname{cat} f \leqq w \operatorname{cat} f \leqq \operatorname{cat} f$ for any map f.

Let $f: X \to Y$, $g: Y \to Z$ be maps. Then it is easily seen that $\operatorname{cat}(gf) \leqq \min\{\operatorname{cat} f, \operatorname{cat} g\}$ and $w \operatorname{cat}(gf) \leqq \min\{w \operatorname{cat} f, w \operatorname{cat} g\}$.

THEOREM 5. *Let $f: X \to Y$ be a map and let $e': Y \to \Omega\Sigma Y$ be the adjoint of $1_{\Sigma Y}$. Then* $\operatorname{conil} f \leqq \Sigma w \operatorname{cat}(e'f)$.

Proof. Suppose $\Sigma w \operatorname{cat}(e'f) \leqq n$. Then $\Sigma(q\Delta e'f) \simeq *$ where $q: (\Omega\Sigma Y)^{n+1} \to \bigwedge_{i=1}^{n+1} \Omega\Sigma Y$ is the projection. Let $c: \Sigma Y \to \bigvee_{i=1}^{n+1} \Sigma Y$ be the commutator map of weight $(n+1)$ for ΣY. Then we can form a map $\bar{c}: Y^{n+1} \to \Omega(\bigvee_{i=1}^{n+1} \Sigma Y)$ such that $\bar{c}\Delta = \tau(c)$ (see [5]). Since $\Sigma(q\Delta e'f) \simeq *$, applying τ we have $\Omega\Sigma(q\Delta)e'f \simeq *$. Consider the following diagram where each square is homotopy-commutative.

$$\begin{tikzcd}
X \arrow[d, "f"] & & \\
Y \arrow[r, "\Delta"] \arrow[d, "e'"] & Y^{n+1} \arrow[r, "q"] \arrow[d, "e'"] & \bigwedge_{i=1}^{n+1} Y \arrow[d, "e'"] \\
\Omega\Sigma Y \arrow[r, "\Omega\Sigma\Delta"'] & \Omega\Sigma(Y^{n+1}) \arrow[r, "\Omega\Sigma q"'] & \Omega\Sigma\left(\bigwedge_{i=1}^{n+1} Y\right)
\end{tikzcd}$$

We have then that $e'q\Delta f \simeq *$. Using Lemmas 4.1_k and 4.2_k of [5], it follows that $\bar{c}\Delta f \simeq *$, that is, $\tau(c)f \simeq *$. Hence $c(\Sigma f) \simeq *$, and hence $\operatorname{conil} f \leqq n$. This proves that $\operatorname{conil} f \leqq \Sigma w \operatorname{cat}(e'f)$.

THEOREM 6. *Let $f: X \to Y$ be a map where Y is an H-space. Then* $\operatorname{cat} f = \operatorname{cat}(e'f)$, $w \operatorname{cat} f = w \operatorname{cat}(e'f)$ *where $e': Y \to \Omega\Sigma Y$ is the adjoint of $1_{\Sigma Y}$.*

Proof. We need only show that $\operatorname{cat} f \leqq \operatorname{cat}(e'f)$, and

$$w \operatorname{cat} f \leqq w \operatorname{cat}(e'f) .$$

Since Y is an H-space, we have a map $r: \Omega\Sigma Y \to Y$ such that $re' \simeq 1_Y$. Then $\operatorname{cat} f = \operatorname{cat}(re'f) \leqq \operatorname{cat}(e'f)$ and $w \operatorname{cat} f = w \operatorname{cat}(re'f) \leqq w \operatorname{cat}(e'f)$.

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