PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 1, 1969

AN APPLICATION OF A NEWTON-LIKE METHOD TO THE EULER-LAGRANGE EQUATION

RICHARD A. TAPIA

It is known that any function which minimizes a functional of the form $J(y) = \int_a^b f(x, y, y')$ and satisfies prescribed boundary values must be a solution of the corresponding Euler-Lagrange equation: $f_3(x, y, y') - \int_a^x f_2(x, y, y') = c$. Let us call any equation of the form: $g(x, y, y') - \int_a^x h(x, y, y') = c$ a generalized Euler-Lagrange equation.

In this paper we propose a Newton-like method and show that this proposed method is general enough to enable us to construct solutions of the generalized Euler-Lagrange equation.

Let X and Y be Banach spaces, Ω an open subset of X and $P: \Omega \rightarrow Y$. By $[X, Y]$ we mean the Banach space of all bounded linear operators with the usual operator norm, by P' the first derivative of P and by P'' the second derivative of P. The class of all functions defined on Ω which have continuous derivatives up to and including order n at each point of Ω is denoted by $C^n(\Omega)$. Distinct elements of $C^n(\Omega)$ may have totally distinct ranges depending on the application. The distinction between Gateaux and Fréchet is unnecessary since the derivatives will be continuous.

2. The weak Newton sequence. Let H and Y be Banach spaces, Ω a nonempty open subset of H and $P: \Omega \rightarrow Y$.

DEFINITION. Given $x_0 \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - [P'(x_n)]^{-1}P(x_n) ,$$

is called the Newton sequence for x_0 (with respect to P).

DEFINITION. Given $x_0, \bar{x} \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - [P'(\bar{x})]^{-1}P(x_n) ,$$

is called the modified Newton sequence for x_0 at $\bar{x}$ (with respect to P). When $x_0 = \bar{x}$ we say simply the modified Newton sequence for x_0.

REMARK. The Newton sequences exist if and only if $P'(\bar{x})$ and $P'(x_n)$ exist and are invertible and $x_n \in \Omega$ for all n.

Let X and Y be Banach spaces, H a closed linear subspace of X, Ω a nonempty open subset of H and $P\colon \Omega \to Y$.

REMARK. It is easy to verify that Ω is open in X if and only if $H = X$.

Let D be the set of all $x \in \Omega$ for which there exists an operator $\Gamma(x)$ satisfying the following conditions:

(2.1)
(i) $\Gamma(x) \in [B_x, X]$, where B_x is a closed linear subspace of Y containing $P(\Omega)$;
(ii) $\Gamma(x)P(\Omega) \subset H$;
(iii) $\Gamma(x)P'(x) = I\colon H \to H$.

REMARK. The fact that $\Gamma(x)$ is defined on $P'(x)(H)$ and $\Gamma(x)P'(x)$ is defined from H into H is a consequence of conditions (i) and (ii). The following lemma shows this to be true.

LEMMA 2.1. *If there exists an operator Γ_0 satisfying the following conditions*:
(a) $\Gamma_0 \in [B_0, X]$, *where B_0 is a closed linear subspace of Y containing $P(\Omega)$*;
(b) $\Gamma_0 P(\Omega) \subset H$;
then for all $x \in \Omega$ we have the following:
(i) $P'(x)(H) \subset B_0$;
(ii) $\Gamma_0 P'(x)(H) \subset H$.

Proof. Assume $x \in \Omega$ and $h \in H$. It follows from (a) since Ω is open in H that for small t

$$\frac{P(x + th) - P(x)}{t} \in B_0 .$$

Therefore $P'(x)(h) \in B_0$ and $P'(x)(H) \subset B_0$. Similarly (a), (b) and the fact that H is closed imply

$$\Gamma_0 P'(x)(h) = \lim_{t \to 0} \left[\frac{\Gamma_0 P(x + th) - \Gamma_0 P(x)}{t} \right] \in H .$$

DEFINITION. If $x \in D$, then $\Gamma(x)$ is called a left inverse for $P'(x)$.

REMARK. If $H = X$ and $B_x = Y$, then this is the usual notion of a left inverse.

Let $\Gamma(x)$ denote a left inverse for $P'(x)$.

DEFINITION. Given $x_0 \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - \Gamma(x_n)P(x_n) ,$$

is called the weak Newton sequence for x_0 (with respect to P).

DEFINITION. Given $x_0, \bar{x} \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - \Gamma(\bar{x})P(x_n) ,$$

is called the weak modified Newton sequence for x_0 at $\bar{x}$ (with respect to P). When $x_0 = \bar{x}$ we say simply the weak modified Newton sequence for x_0.

REMARK. The weak Newton sequences exist if and only if $\Gamma(\bar{x})$ and $\Gamma(x_n)$ exist and $x_n \in \Omega$ for all n.

LEMMA 2.2. *If* $x \in \Omega$ *and* $[P'(x)]^{-1} \in [Y, H]$, *then for any* $\Gamma(x)$ *satisfying* (2.1) *we have*:
(i) $B_x = Y$;
(ii) $\Gamma(x)(Y) = H$;
(iii) $\Gamma(x) = [P'(x)]^{-1}$.

Proof. Since $P'(x)$ is onto, $Y = P'(x)(H)$. By Lemma 2.1 $P'(x)(H) \subset B_x$, therefore $Y = B_x$. Also $\Gamma(x)y = \Gamma(x)P'(x)[P'(x)^{-1}y] = P'(x)^{-1}y$ for all $y \in Y$, therefore $\Gamma(x) = [P'(x)]^{-1}$.

REMARK. In general there are many weak (modified) Newton sequences for a particular point; however, by Lemma 2.2 if the (modified) Newton sequence exists, then any weak (modified) Newton sequence coincides with it.

EXAMPLE. Let $P: R^1 \to R^2$ be given by $P(x) = (3x - 3/2, 2x - 1)$ hence $P'(x) = (3, 2)$. If $\Gamma_0: R^2 \to R^1$ is given by

$$\Gamma_0(x_1, x_2) = \left(a, \frac{1 - 3a}{2}\right) \cdot (x_1, x_2)$$

for any real a, then $\Gamma_0 P(x) = x - 1/2$, $\Gamma_0 P'(x) = 1$, the identity map in R^1, and the weak (modified) Newton sequence for x_0 is given by

$$x_{n+1} = x_n - \Gamma_0 P(x_n) = \frac{1}{2}$$

for all n. Notice that $P(1/2) = (0, 0)$.

REMARK. In this example there are an uncountable number of left inverses for $P'(x)$; also the (modified) Newton sequence for x_0 does not exist.

3. **Convergence theorems.** As before let X and Y be Banach spaces, H a closed linear subspace of X, Ω a nonempty open subset of H and $P: \Omega \rightarrow Y$. If for each $x \in \Omega$ there exists an operator $\Gamma(x)$ satisfying conditions 2.1 and we let $B_0 = \bigcap_{x \in \Omega} B_x$, where B_x is the domain of $\Gamma(x)$, then B_0 is a closed linear subspace of Y containing $P(\Omega)$; consequently by restricting $\Gamma(x)$ to B_0 we may consider $\Gamma: \Omega \rightarrow [B_0, X]$.

THEOREM 3.1. *If*

(i) *there exists* $x^* \in \Omega$ *such that* $P(x^*) = 0$,

(ii) *for each* $x \in \Omega$ *there exists* $\Gamma(x)$ *satisfying conditions* (2.1) *and* $\Gamma: \Omega \rightarrow [B_0, X] \in C^0(\Omega)$

then, if $P \in C^1(\Omega)$, *given* $0 < \alpha < 1$ *there exists a neighborhood of* x^* *contained in* Ω *such that for any* 2 *points* x_0 *and* $\bar{x}$ *in this neighborhood the weak modified Newton sequence for* x_0 *at* $\bar{x}$ *exists and converges to* x^*; *we also have*

$$\| x^* - x_n \| \leqq \frac{\alpha^n}{1 - \alpha} \| x_1 - x_0 \| .$$

Furthermore, if $P \in C^2(\Omega)$, *then there exists a neighborhood of* x^* *contained in* Ω *such that if* x_0 *is any point in this neighborhood a weak Newton sequence for* x_0 *exists and converges quadratically to* x^*, *i.e., there exists a constant* M *such that*

$$\| x^* - x_{n+1} \| \leqq M \| x^* - x_n \|^2 .$$

Proof. For $x \in \Omega$ let $T(x) = I - \Gamma(x)P'(x^*)$. Given $0<\alpha<1$ there exists $\delta > 0$ such that $\{x \mid \| x - x^* \| \leqq \delta\} \subset \Omega$ and

$$\| T(x) \| = \| T(x) - T(x^*) \| < \alpha$$

whenever $\| x - x^* \| \leqq \delta$. Now if $\| \bar{x} - x^* \| \leqq \delta$ and $S = I - \Gamma(\bar{x})P$, then

(i) $S(x^*) = x^*$ and

(ii) $\| S'(x^*) \| < \alpha$.

Let $\| S'(x^*) \| = \alpha_0$, then there exists $0 < \delta_0 \leqq \delta$ such that

$$\| S'(x) - S'(x^*) \| \leqq \alpha - \alpha_0$$

whenever $x \in \Omega_0 = \{x \mid \| x - x^* \| \leqq \delta_0\}$. Clearly Ω_0 is closed, convex and contained in Ω. Also

(i)' $\| S'(x) \| \leqq \alpha$ whenever $x \in \Omega_0$ and

(ii)′ $S(\Omega_0) \subset \Omega_0$;
since if $x \in \Omega_0$,

$$\begin{aligned} \| S(x) - x^* \| &= \| S(x) - S(x^*) \| \\ &\leqq \| S'(x + \theta(x^* - x)) \| \, \| x - x^* \| \qquad 0 < \theta < 1 \\ &\leqq \alpha \| x - x^* \| \leqq \delta_0 . \end{aligned}$$

The first part of this theorem now follows from a well-known fixed point theorem [4, 661]. The latter part of the theorem is a consequence of the following Banach space inequality [3, pp. 162–163].

$$\| P(x + \Delta x) - P(x) - P'(x)(\Delta x) \| \leqq \tfrac{1}{2} \| P''(x + \theta\Delta x) \| \, \| \Delta x \|^2 , \tag{3.1}$$

$0 < \theta < 1$. There clearly exists K and $\delta > 0$ such that $\| P''(x) \| \leqq K$ and $\| \Gamma(x) \| \leqq K$, whenever $\| x^* - x \| \leqq \delta$. If $\| x^* - x \| \leqq \delta$, then from (3.1) with $\Delta x = x^* - x$ we obtain

$$\| x^* - [x - \Gamma(x)P(x)] \| \leqq M \| x^* - x \|^2 \tag{3.2}$$

where $M = \frac{1}{2}K^2$. Choose $0 < \delta_0 < \min(M^{-1}, \delta)$ and such that

$$\Omega_0 = \{x \mid \| x^* - x \| \leqq \delta_0\} \subset \Omega .$$

We now show if $X_n \in \Omega_0$, then

$$\begin{aligned} \| x^* - x_{n+1} \| &\leqq M \| x^* - x_n \|^2 \\ &\leqq \| x^* - x_0 \| (M \| x^* - x_0 \|)^{2^{n+1}-1} \leqq \delta_0 ; \end{aligned} \tag{3.3}$$

consequently $x_{n+1} \in \Omega_0$. For $n = 0$ inequality (3.3) is just (3.2) with $x = x_0$. If we assume (3.3) holds for $n < k$, then a direct application of (3.2) shows (3.3) holds for $n = k$, and consequently for all n. This proves the theorem.

Consider $\Gamma_0 \colon B_0 \to X$, where B_0 is a closed linear subspace of Y containing $P(\Omega)$, satisfying:

$$\begin{aligned} &(\mathrm{i}) \quad \Gamma_0 \in [B_0, X] ; \\ &(\mathrm{ii}) \quad \Gamma_0 P(\Omega) \subset H . \end{aligned} \tag{3.4}$$

COROLLARY 3.1. *Given $x_0 \in \Omega$, if there exists*

(i) *$x^* \in \Omega$ such that $P(x^*) = 0$, and*

(ii) *Γ_0 satisfying* (3.4) *and such that $\Gamma_0 P'(x_0) \colon H \to H$ is invertible,*

then if x_0 is sufficiently near x^* and $P \in C^2(\Omega)$ a weak (modified) Newton sequence for x_0 exists and converges to x^*. Furthermore a left inverse for $P'(x)$ is given by $\Gamma(x) = [\Gamma_0 P'(x)]^{-1}\Gamma_0$.

Proof. Use Theorem 3.1 on $Q = \Gamma_0 P$.

4. A general problem. Given an interval $[a, b]$ let

$$
\begin{aligned}
&X = \{y \in C^0[a, b] \mid y\colon [a, b] \to R^1\}\,, \\
&H = \left\{h \in X \,\middle|\, \int_a^b h = 0\right\}, \\
&Y = \{y \in C^1[a, b] \mid y\colon [a, b] \to R^1\}\,, \\
(4.1)\qquad &K = \{y \in Y \mid y(a) = \mu_1 \text{ and } y(b) = \mu_2\}\,, \\
&B = [a, b] \times R^2\,, \\
&\| y \|_X = \sup\{| y(t) | \mid t \in [a, b]\} \text{ for } y \in X, \text{ and} \\
&\| y \|_Y = \sup\{| y(t) |, | y'(t) | \mid t \in [a, b]\} \text{ for } y \in Y\,.
\end{aligned}
$$

REMARK. If X, Y and H are given by (4.1), then X and Y are Banach spaces and H is a closed linear subspace of X.

Given continuous $\varphi\colon B \to R^1$ and $y_0 \in K$ define

$$\bar{Q}\colon Y \to X\,, \quad \text{and} \quad Q\colon X \to X$$

as follows

$$(4.2)\qquad \bar{Q}(y)(x) = \varphi(x, y(x), y'(x)) \quad \text{for} \quad y \in Y\,;$$

$$(4.3)\qquad Q(y) = \bar{Q}\left(y_0 + \int_a^x y\right) \quad \text{for} \quad y \in X\,.$$

It is clear that if $y \in X$, then $y_0 + \int_a^x y \in Y$. Assume Q satisfies the following two conditions:

$$(4.4)\qquad \begin{aligned} &(\mathrm{i})\quad Q \in C^2[X]\,; \\ &(\mathrm{ii})\quad [Q'(0)]^{-1} \in [X, X]\,. \end{aligned}$$

Define $P\colon H \to X$ by

$$(4.5)\qquad P(h) = Q(h) - \frac{\int_a^b \Gamma_0 Q(h)}{\int_a^b \Gamma_0(1)} \quad \text{for} \quad h \in H\,,$$

where $\Gamma_0 = [Q'(0)]^{-1}$ and 1 is the constant function 1.

LEMMA 4.1. *If $\bar{Q}$ is given by* (4.2) *and P by* (4.5), *then the following two problems are equivalent*:

(4.6) *Problem* I. *Find $y \in K$ such that $\bar{Q}(y)$ is constant*;
Problem II. *Find $h \in H$ such that $P(h) = 0$*.

Proof. If $y \in K$ and $\bar{Q}(y)$ is constant, then $h = y' - y_0' \in H$ and

$Q(h) = \bar{Q}\left(y_0 + \int_a^x (y' - y_0')\right)$ is constant; therefore $P(h) = 0$. Now if $h \in H$ and $P(h) = 0$, then $Q(h) = \bar{Q}\left(y_0 + \int_a^x h\right)$ is constant; therefore $y = y_0 + \int_a^x h \in K$ and $\bar{Q}(y)$ is constant.

LEMMA 4.2. *If Γ_0 and P are given by* (4.5), *then Γ_0 is a left inverse for $P'(0)$.*

Proof. We show $\Gamma(0) = \Gamma_0$ satisfies (2.1) for $x = 0$. If $B_0 = X$ and $\Omega = H$, then (i) of (2.1) holds. If $h \in H$, then from (4.5) we obtain,

$$\int_a^b \Gamma_0 P(h) = 0$$

and (ii) of (2.1) holds. By differentiating (4.5) at the origin we see that

$$\Gamma_0 P'(0)(h) = h \quad \text{for} \quad h \in H .$$

Therefore $\Gamma_0 P'(0) = I\colon H \to H$ and the lemma is proved.

Let $y^* \in Y$ be a solution of Problem I and $h^* \in X$ the corresponding solution of Problem II.

LEMMA 4.3. *For given y_0 in* (4.3)

$$\| h^* \|_X \leqq \| y_0 - y^* \|_Y .$$

Proof. The proof follows from (4.1) and Lemma 4.1.

REMARK. If Problem I has a solution and the given y_0 in (4.3) is sufficiently near this solution, then by Lemmas 4.1, 4.2, 4.3 and Corollary 3.1 with $\Gamma_0 = [Q'(0)]^{-1}$, $x_0 = 0$, $\Omega = H$, $Y = X$ and H, X and P given by (4.1) and (4.5) both the weak Newton and weak modified Newton sequences for x_0 exist and can be used to obtain this solution. In addition the weak modified Newton sequence can still be used if we only have $Q \in C^1[H]$.

5. A variation of the weak Newton sequence. If Γ_0 and P are given by (4.5) and h_n is the n^{th} term in the weak Newton sequence for $h_0 = 0$, then by Corollary 3.1 a left inverse for $P'(h_n)$ is given by

$$\Gamma(h_n) = [\Gamma_0 P'(h_n)]^{-1} \Gamma_0 .$$

This operator may be difficult to evaluate; we therefore consider the following variation of the weak Newton sequence for Problem II.

For Q given by (4.3), assume the following conditions hold:

$$\text{(5.1)}\qquad \begin{aligned}&\text{(i)}\quad Q \in C^2[X];\\ &\text{(ii)}\quad [Q'(x)]^{-1} \in [X, X] \text{ for all } x \in X.\end{aligned}$$

Inherent in the above assumption is the requirement that a procedure for evaluating $[Q'(x)]^{-1}$ is known. Let

$$\text{(5.2)}\qquad \Gamma(x) = [Q'(x)]^{-1} \text{ for } x \in X$$

and define $h_n \in H$ and $P_n\colon H \to X$ recursively by

$$\text{(5.3)}\qquad \begin{aligned}&h_0 = 0\\ &P_0(h) = Q(h) - \frac{\int_a^b \Gamma(h_0)Q(h)}{\int_a^b \Gamma(h_0)(1)} \quad \text{for} \quad h \in H, \text{ and}\\ &h_{n+1} = h_n - \Gamma(h_n)P_n(h_n)\,,\\ &P_{n+1}(h) = Q(h) - \frac{\int_a^b \Gamma(h_{n+1})Q(h)}{\int_a^b \Gamma(h_{n+1})(1)}\,, \quad \text{for} \quad n = 0, 1, \cdots.\end{aligned}$$

LEMMA 5.1. *If $\Gamma(h_n)$ and $P_n(h_n)$ are given by* (5.2) *and* (5.3), *then $\Gamma(h_n)$ is a left inverse for $P_n'(h_n)$.*

Proof. The proof of this lemma is the same as the proof of Lemma 4.2.

THEOREM 5.1. *If* (5.1) *holds and the given y_0 in* (4.3) *is sufficiently close to a solution of Problem* I, *then the sequence $\{h_n\}_0^\infty$ given by* (5.3) *will converge quadratically to the corresponding solution of Problem* II.

Proof. The proof of this theorem is essentially the same as the latter part of the proof of Theorem 3.1.

LEMMA 5.2. *For y_0 used in* (4.3) *and $\{h_n\}_0^\infty$ given by* (5.3), *Problem* I *is equivalent to the following problem*:

Problem III. *Find $y_0 \in K$ such that the sequence $\{h_n\}_0^\infty$ converges.*

Proof. If y is a solution to Problem I then by Theorem 5.1 we need only pick $y_0 \in K$ near y. Now assume $\{h_n\}_0^\infty$ converges to h, then $\Gamma(h_n)P_n(h_n) \to 0$ and since $\Gamma^{-1}(h) = Q'(h)$ exists we have $P_n(h_n) \to 0$ or $Q(h)$ is constant. Also, since H is closed, $h \in H$; by Lemma 4.1 $y =$

$y_0 + \int_a^x h$ solves Problem 1.

REMARK. We have shown that whenever the sequence $\{h_n\}_0^\infty$ given by (5.3) converges, it converges quadratically to a solution of Problem II and consequently gives a solution to Problem I.

6. **The generalized Euler-Lagrange equation.** Let X, H, Y, K and B be given by (4.1). In the calculus of variations one is interested in finding $y \in K$ such that for all $x \in [a, b]$

$$f_3(x, y(x), y'(x)) - \int_a^x f_2(x, y(x), y'(x)) = c \,, \tag{6.1}$$

where $f\colon B \to R^1$, f_i denotes the i^{th} partial derivative of f and c is an unknown constant. Historically equation (6.1) has been called the Euler-Lagrange equation.

For $g, h\colon B \to R^1$, we would like to find $y \in K$ such that for all $x \in [a, b]$

$$g(x, y(x), y'(x)) - \int_a^x h(x, y(x), y'(x)) = c \,. \tag{6.2}$$

We are therefore interested in solving Problem I (4.6) with $\bar{Q}\colon Y \to X$ given by:

$$\bar{Q}(y)(x) = g(x, y(x), y'(x)) - \int_a^x h(x, y(x), y'(x)) \,. \tag{6.3}$$

Since equation (6.1) is a special case of equation (6.2), we call (6.2) the generalized Euler-Lagrange equation.

Given $f_{i,j} \in X$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, 5$, define the integral operator $L\colon X \to X$ as follows: for $u \in X$ let

$$\begin{aligned} \mathrm{L}(u)(x) = \sum_{i=1}^{n} \Big[& f_{i,1}(x) \int_a^x f_{i,2}(y) u(y) dy \\ & + f_{i,3}(x) \int_a^x \Big(f_{i,4}(y) \int_a^y f_{i,5}(t) u(t) dt \Big) dy \Big] \,. \end{aligned} \tag{6.4}$$

THEOREM 6.1. *If L is given by* (6.4) *and λ is any constant, then*
(i) $L \in [X, X]$,
(ii) *for all $u \in X$ the series*

$$\| u \| + \| \lambda L(u) \| + \cdots + \| \lambda^n L^n(u) \| + \cdots$$

is convergent.
Furthermore,
(iii) *for all $\varphi \in X$ the equation*

$$f = \varphi + \lambda L(f) , \tag{6.5}$$

has a unique solution in X which can be obtained by iterating (6.5) *beginning with an arbitrary element of X; finally*

(6.6) (iv) *the operator $T = I - \lambda L$ has a continuous inverse which can be evaluated by iteration.*

Proof. The proof of (i) is immediate. Since (iii) proceeds directly from (ii) and (iv) follows from (i) and (iii) we will only prove (ii).

For $u \in X$ let $K(u)(x) = \int_a^x u$. Then there exists a constant B such that for $x \in [a, b]$

$$| L(u)(x) | \leqq B(K + K^2)(| u |)(x) \leqq B \| u \| (K + K^2)(1)(x) .$$

Assume for $1 \leqq n \leqq k$

$$| L^n(u)(x) | \leqq B^n(K + K^2)^n(| u |)(x) \leqq B^n \| u \| (K + K^2)^n(1)(x) , \tag{6.7}$$

then

$$\begin{aligned} | L^{k+1}(u)(x) | &= | L^k(L(u))(x) | \leqq B^k(K + K^2)^k(| L(u) |)(x) \\ &\leqq B^{k+1}(K + K^2)^{k+1}(| u |)(x) \leqq B^{k+1} \| u \| (K + K^2)^{k+1}(1)(x) \end{aligned}$$

and by induction (6.7) holds for all n.

If we let $M = \max(1, \| u \|, | \lambda | B)$ and denote $K^n(1)(x)$ by K^n, then

$$\begin{aligned} \sum_{n=0}^{\infty} | \lambda^n L^n(u)(x) | &\leqq \sum_{n=0}^{\infty} M^{n+1}(K + K^2)^n \leqq \sum_{n=0}^{\infty} \sum_{j=0}^{n} M^{n+j+1}\binom{n}{j} K^{n+j} \\ &= \sum_{m=0}^{\infty} A_m M^{m+1} \frac{(x - a)^m}{m!} , \end{aligned} \tag{6.8}$$

where $A_m = \sum_{i=0}^{[m/2]} \binom{m-i}{i}$ and $[x]$ denotes the largest integer less than or equal to x. The first inequality in (6.8) follows from (6.7) and the second inequality is a direct application of the binomial theorem. The equality in (6.8) is justified by observing that the coefficients of K^m for arbitrary but fixed m are of the form $M^{n+j+1}\binom{n}{j}$ where $j \leqq n$ and $n + j = m$ and also that $K^m = (x - a)^m/m!$. It is not difficult to show that A_m satisfies the difference equation $A_{m+1} = A_m + A_{m-1}$; consequently the radius of convergence of the last series in (6.8) is infinite. This proves (ii) and the theorem follows.

LEMMA 6.1. *For $g, h \in C^0(B)$ let Q be given by* (4.3) *and* (6.3). *If $g, h \in C^n(B)$ then $Q \in C^n(X)$, for $n = 0, 1, 2$.*

Proof. The proof is straightforward although somewhat lengthy [6].

If X, K and B are given by (4.1) and $y_0 \in K$, then for continuous $g: B \to R^1$ define $\bar{g}: X \to X$ as follows:

$$\bar{g}(y)(x) = g\left(x, y_0(x) + \int_a^x y(t), y_0'(x) + y(x)\right)$$

for $y \in X$ and $x \in [a, b]$.

THEOREM 6.2. *For $g, h: B \to R^1$, if*
(a) *$g, h \in C^2(B)$, and*
(b) *$\bar{g}_3(y) \neq 0$ for $y \in X$*
and if $Q: X \to X$ is given by (4.3) *and* (6.3) *then*
(i) *$Q \in C^2[X]$,*
(ii) *$[Q'(y)]^{-1} \in [X, X]$ for all $y \in X$ and can be evaluated by iteration.*

Proof. Part (i) follows from Lemma 6.1. A direct calculation shows that

$$Q'(y)(\eta) = \bar{g}_3(y)\eta + \bar{g}_2(y)\int_a^x \eta - \int_a^x \left(\bar{h}_2(y)\int_a^x \eta\right) - \int_a^x \bar{h}_3(y)\eta ,$$

for $y, \eta \in X$. The subscripts on $\bar{g}$ and $\bar{h}$ denote partial derivatives. If we let

$$(6.9) \qquad \begin{aligned} &f = \frac{1}{\bar{g}_3(y)} \\ &f_{11} = f \cdot \bar{g}_2(y), f_{12} = 1, f_{13} = -f, f_{14} = \bar{h}_2(y), f_{15} = 1 , \\ &f_{21} = -f, f_{22} = \bar{h}_3(y) \quad \text{and} \quad f_{23} = f_{24} = f_{25} = 0 , \quad \text{then} \end{aligned}$$

$$T(\eta) = \eta + L(\eta) = f \cdot Q'(y)(\eta)$$

where $y, \eta \in X$ and $L: X \to X$ is given by (6.4) and (6.9) with $n = 2$ and $T: X \to X$ is given by (6.6) with $\lambda = -1$. Given $\varphi, y \in X$ constructing $\eta = [Q'(y)]^{-1}(\varphi)$ is equivalent to solving

$$T(\eta) = f\varphi \quad \text{for} \quad \eta \in X ,$$

hence $\eta = T^{-1}(f\varphi)$ and according to Theorem 6.1, η is given by

$$\eta = f\varphi - L(f\varphi) + L^2(f\varphi) - L^3(f\varphi) + \cdots .$$

REMARKS. It follows that the theory developed in § 1 through § 5 can be used to solve the generalized Euler-Lagrange equation. In the calculus of variations condition (b) above is called the strengthened Legendre condition.

Theorem 6.2 includes the problem of finding $y \in K$ such that

$$y'' = h(x, y, y') \quad \text{where} \quad h \in C^2(B) . \tag{6.10}$$

If $g: B \rightarrow R^1$ is defined by

$$g(x, y, z) = z, \quad \text{then} \quad \bar{g}_3(y) = 1 \neq 0$$

and $g \in C^2(B)$, therefore Theorem 6.2 holds. Now if

$$y' - \int_a^x h(x, y, y') = \text{constant} , \tag{6.11}$$

then since two of the three terms in (6.11) are differentiable, the third must also be, giving (6.10).

The author would like to thank the referee for many helpful comments and suggestions.

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Received February 12, 1968. The research on this paper was sponsored by the Office of Naval Research under Contract NONR 233(76). Reproduction in whole or in part is permitted for any purpose of the United States Government; and represents a part of the author's doctoral dissertation directed by Professor C. B. Tompkins.

UNIVERSITY OF CALIFORNIA, LOS ANGELES
AND
MATHEMATICS RESEARCH CENTER, UNIVESITY OF WISCONSIN