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ON KRULL OVERRINGS OF AN AFFINE RING

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By an overring of an integral domain A we mean a ring which contains A and is contained in the quotient field of A. We consider the following question. If D is a Krull overring of an affine ring A is D necessarily Noetherian? Our main result is an affirmative answer to this question when A is a normal affine ring of dimension two defined over a field or pseudogeometric Dedekind domain such that each localization of A has torsion class group.

We recall that an integral domain J is called a *Krull ring* if J is an intersection of rank one discrete valuation rings, say $J = \bigcap_\alpha V_\alpha$, such that each nonzero element of J is a nonunit in only finitely many of the V_α. One may assume that each V_α is an overring of J and is irredundant in the representation $J = \bigcap_\alpha V_\alpha$. In this case each V_α is centered on a minimal prime (prime of height one) of J and if V_α has center P_α on J, then $J_{P_\alpha} = V_\alpha$. The set $\{V_\alpha\} = \{J_{P_\alpha}\}$ is called the set of *essential valuation rings* for J. We use the notation $E(J)$ to denote the set of essential valuation rings of the Krull ring J.

A one dimensional Krull ring is a Dedekind domain and hence is Noetherian. There exist non-Noetherian 3 dimensional Krull rings, an example being given by Nagata [6, p. 207] who showed that the derived normal ring of a 3 dimensional local domain need not be Noetherian. Whether a 2 dimensional Krull ring is necessarily Noetherian remains open[1]. Since the derived normal ring of a 2 dimensional Noetherian domain is again Noetherian one can not hope to construct non-Noetherian 2 dimensional Krull rings by a method similar to Nagata's. Our results here show that in certain special cases 2 dimensional Krull rings are Noetherian. In fact, the original motivation for our work was to determine if each Krull overring of $Z[X]$ (Z the ring of integers and X an indeterminate over Z) is Noetherian, a question brought to our attention by Jack Ohm. We are grateful to Ohm for several helpful conversations concerning this topic.

2. We will consistently use A to denote a normal affine ring of dimension 2 defined over a field or pseudogeometric Dedekind domain. We will further assume that each localization R of A has torsion class

[1] An exercise in Bourbaki [3, p. 83] outlines a method for constructing a two dimensional Krull ring which is asserted not to be Noetherian. However in [5] an argument is given to the effect that the Bourbaki construction must necessarily yield a Noetherian Krull ring. Recently Paul Eakin has constructed a non-Noetherian 2 dimensional Krull ring.

group. This, of course, is equivalent to the assumption that each minimal prime of R is the radical of a principal ideal.

Our first results concern Krull overrings of a localization of A. Let R be a localization of A. R has dimension either one or two and if R has dimension one then R is a rank one discrete valuation ring and has no nontrivial overrings. We assume therefore that R is of dimension two with maximal ideal M. Let D be a Krull overring of R. If V is an essential valuation ring for D then V either has center M on R or else V is centered on a minimal prime P of R. In the latter case $R_P \subseteq V$, and since R_P is also a rank one discrete valuation ring we have $R_P = V$ and $V \in E(R)$. Thus $E(D) - E(R)$ consists precisely of the essential valuation rings of D having center M on R and the finiteness condition in the definition of a Krull ring insures that $E(D) - E(R)$ is a finite set.

If V is a valuation overring of R centered on M we recall that the R-dimension of V is defined to be the transcendence degree over R/M of the residue field of V. (Here we are using the canonical embedding of R/M in the residue field of V). Since R is two dimensional and Noetherian each such V has R-dimension either zero on one [1, p. 328]. Moreover, if V has R-dimension zero then V is necessarily centered on a maximal ideal of any domain between R and V. Let $\{V_i\}_{i=1}^n$ be the subset of $E(D) - E(R)$ consisting of those elements of $E(D) - E(R)$ which have R-dimension zero and let D' be the Krull ring having $E(D) - \{V_i\}$ as its set of essential valuation rings. We now observe that to show D is Noetherian it will suffice to show that D' is Noetherian. This is a consequence of the following proposition.

PROPOSITION 1. *Let J be a Krull ring and let V be an essential valuation ring for J whose center P on J is a maximal ideal. Let J' be the Krull overring of J having $E(J) - \{V\}$ as its set of essential valuation rings. If J' is Noetherian, then J is Noetherian.*

Proof. We note that J' is the P-transform of J as defined by Nagata in [7, p. 58]. Also $PJ' \cap J$ properly contains P so that $PJ' = J'$. Hence there is a one-to-one correspondence between the prime ideals of J' and the prime ideals of J excluding P where a prime ideal Q' of J' is associated with $Q' \cap J = Q$ [7, p. 58] or [8, p. 198]. We choose $\{x_1, \cdots, x_n\} = X \subseteq P$ so that $XJ' = J'$. We may also assume that $XJ_P = PJ_P$. Then $XJ = P$ since $XJ_Q = PJ_Q = J_Q$ for each maximal ideal Q of J distinct from P. Hence P is finitely generated[2]. Let Q be a prime of J distinct from P with Q' being the unique prime of

[2] We have in fact established that P is invertible, for P is finitely generated and localized at any maximal ideal P is principal.

J' such that $Q' \cap J = Q$. By assumption Q' is finitely generated, say $\{y_1, \cdots, y_m\} = Y$ generates Q'. There exists an integer t such that $YP^t \subseteq J$. Hence $YJ \cdot P^t = B$ is a finitely generated ideal of J such that $BJ' = Q'$. By enlarging B if necessary we may assume that $B \not\subseteq P$. Thus $BJ_P = QJ_P = J_P$. If N is a maximal ideal of J distinct from P and N' is the unique maximal ideal of J' with $N' \cap J = N$ then $J_N = J'_{N'}$. Hence $QJ_N = Q'J'_{N'} = BJ'_{N'}$. It follows that $B = Q$[9, p. 94]. We have thus shown that each prime ideal of J is finitely generated and hence that J is Noetherian. This completes the proof of Proposition 1.

We now construct a normal Noetherian ring R' such that R' is finitely generated over R and $E(D') \subseteq E(R')$. Let $\{W_i\}_{i=1}^m = E(D') - E(R)$ and let T_i be the maximal ideal of W_i. Since W_i is a quotient ring of D' we see that $D'/T_i \cap D'$ has quotient field W_i/T_i. By assumption W_i/T_i is transcendental over $R/T_i \cap R = R/M$. We choose a_i in D' such that the residue of a_i in W_i/T_i is transcendental over R/M. Then W_i is not centered on a maximal ideal of $R[a_i]$ so that W_i is necessarily an essential valuation ring for R', the integral closure of $R[a_1, \cdots, a_m]$. Since R' is a finite $R[a_1, \cdots, a_m]$-module we conclude that R' is again a quotient ring of a normal affine ring of dimension two defined over a field or pseudogeometric Dedekind domain. Moreover $E(D') \subseteq E(R')$.

We proceed to show that D' is Noetherian. If J is a Krull ring let $C(J)$ denote the class group of J and let $C_1(J)$ be the torsion free quotient group $C(J)/C_2(J)$ where $C_2(J)$ is the torsion subgroup of $C(J)$. As Claborn observed in [4, p. 220] if J and J' are Krull rings with $E(J') \subseteq E(J)$ then $C(J')$ is a homomorphic image of $C(J)$ and the kernel of this canonical homomorphism is generated by the classes of all minimal primes P of J such that $J_P \in E(J) - E(J')$. Since $C_1(R)$ is trivial[3] and $E(R') - E(R)$ is a finite set we see that $C_1(R')$ is finitely generated. Hence $C_1(R')$ is free abelian on a finite set of generators. The canonical homomorphism $\varphi: C(R') \to C(D')$ enduces an onto homomorphism $\varphi_1: C_1(R') \to C_1(D')$. Let $\{P_i\}_{i=1}^k$ be minimal primes of R' whose equivalence classes in $C_1(R')$ generate the kernel of φ_1. Let $Q = \bigcap_{i=1}^k P_i$ and let S be the Q-transform of R'. Since R' is a quotient ring of a normal affine ring of absolute dimension two, Nagata's results in [7] and [8] imply that S is finitely generated over R'. Moreover the canonical homomorphism $\psi_1: C_1(S) \to C_1(D')$ is an isomorphism. This means that each minimal prime P of S such that $S_P \in E(S) - E(D')$ is the radical of a principal ideal which in turn implies that D' is a quotient ring of S. Since S is Noetherian we conclude that D' is Noetherian[4]. We summarize the results of this section in the following theorem.

[3] It would suffice here to assume that $C_1(R)$ is finitely generated.

[4] We have actually shown that D' is a quotient ring of a normal affine ring.

THEOREM 2. *Let R be a localization of a normal affine ring A, where A is defined over a field on pseudogeometric Dedekind domain and has dimension two. If the class group of R is a torsion group, or more generally if $C_1(R)$ is finitely generated, and if D is a Krull overring of R then D is Noetherian.*

3. We turn now to the consideration of an arbitrary Krull overring D of A. Our main result is the following.

THEOREM 3. *Let A be a normal affine ring of dimension two defined over a field or psuedogeometric Dedekind domain and assume that each localization of A has torsion class group. If D is a Krull overring of A, then D is Noetherian.*[5]

Proof. Let P' be a prime ideal of D and let $P = P' \cap A$. If $S = A - P$ then $A_S \subseteq D_S$ and by Theorem 2 D_S is a Noetherian domain. Let X be a finite set of generators for P and let Y be a finite subset of D such that $YD_S = P'D_S$. If P is a maximal ideal of A we observe that $X \cup Y = Z$ is a finite basis for P'. For this purpose it will suffice to show that $ZD_{M'} = P'D_{M'}$ for each maximal ideal M' of D. If $P \not\subseteq M'$ then $X \not\subseteq M'$ and $ZD_{M'} = P'D_{M'} = D_{M'}$. However if $P \subseteq M'$ then $D_S \subseteq D_{M'}$. Hence $P'D_{M'} = YD_{M'} = ZD_{M'}$. We conclude that P' is finitely generated when $P' \cap A = P$ is a maximal ideal of A.

Consider now the case when P is a minimal prime of A. We have $A_P \subseteq D_{P'}$ and A_P is a discrete rank one valuation ring. Hence $A_P = D_{P'}$ and $D_{P'}$ is an essential valuation ring for D. Now the nonzero elements of P are positive in only finitely many of the essential valuation rings for D. Let $\{V_i\}_{i=1}^n$ be the essential valuation rings for D distinct from $D_{P'}$ in which the elements of P are positive. (Of course the set $\{V_i\}$ may be empty). Each V_i is centered on a maximal ideal M_i of A. Let $S_i = A - M_i$. Then $A_{S_i} \subseteq D_{S_i}$ and again by Theorem 2, D_{S_i} is a Noetherian domain. Let Y_i be a finite subset of D such that $Y_i D_{S_i} = P'D_{S_i}$ and again let X be a finite basis for P. In this case we set $Z = \bigcup_{i=1}^n Y_i \cup X$. If M' is a maximal ideal of D and $P \not\subseteq M'$ then as before $X \not\subseteq M'$ and $ZD_{M'} = P'D_{M'} = D_{M'}$. If $P \subseteq M'$ and $M' \cap A = M_i$ then $ZD_{M'} = Y_j D_{M'} = P'D_{M'}$. In the remaining case let $M = M' \cap A$ and $S = A - M$. We have $A_S \subseteq D_S$ and $E(D_S) \subseteq E(A_S)$. Moreover $C(A_S)$ is a torsion group so that D_S is a quotient ring of A_S[4, p. 219]. Hence $P'D_S = PD_S = ZD_S$, and $P'D_{M'} = ZD_{M'}$. We conclude that $P' = ZD$ and hence that D is Noetherian.

[5] That not every Krull overring of a 3 dimensional normal affine ring need be Noetherian has recently been shown in joint work of the author and Paul Eakin.

COROLLARY 4. *If A is a polynomial ring in two variables over a field or more generally a polynomial ring in one variable over a pseudogeometric Dedekind domain, then each Krull overring of A is Noetherian.*

Proof. We need only observe that each localization of A has torsion class group. If $A = D[X]$ with D a Dedekind domain and if P is a prime of height 2 in A with $Q = P \cap D$ then A_P is a quotient ring of the unique factorization domain $D_Q[X]$. Thus each localization of A has torsion class group.

Added in proof. In a paper submitted to Proc. Amer. Math. Soc., the author has now shown that each Krull overring of a 2-dimensional Noetherian domain is again Noetherian.

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