PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 1, 1969

PRIMARY ABELIAN GROUPS MODULO FINITE GROUPS

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Let $\mathscr{A}$ be the category of Abelian groups, $\mathscr{F}$ the Sèrre class of finite Abelian groups, and form the quotient category $\mathscr{A}/\mathscr{F}$. The purpose of this paper is to find a complete set of invariants for direct sums of countable reduced p-groups, such groups being considered as objects of the category $\mathscr{A}/\mathscr{F}$. Specifically, it will be shown that two direct sums of countable reduced p-groups G and H are isomorphic in $\mathscr{A}/\mathscr{F}$ if and only if

$f_G(\alpha) = f_H(\alpha)$ **for almost all ordinal numbers** α,

and

$f_G(\alpha) \neq f_H(\alpha)$ **implies** $\max(f_G(\alpha), f_H(\alpha)) < \aleph_0$.

The objects of the quotient category $\mathscr{A}/\mathscr{L}$ where $\mathscr{L}$ is any Serre class of $\mathscr{A}$ are just the objects of $\mathscr{A}$. A description of $\mathrm{Hom}_{\mathscr{A}/\mathscr{L}}(G, H)$ for objects G and H of $\mathscr{A}/\mathscr{L}$ can be found in [5]. Such a description is omitted here since two Abelian groups G and H are isomorphic in $\mathscr{A}/\mathscr{L}$ if and only if there exist subgroups S and A of G, and T and B of H such that $S/A \simeq T/B$ in $\mathscr{A}$ and G/S, A, H/T, $B \in \mathscr{L}$. (See 2.3 and 2.4 in [5].) Consequently all computations can be made in $\mathscr{A}$ the category of Abelian groups. In particular the words group and homomorphism will always mean Abelian group and $\mathscr{A}$-homomorphism, respectively. For a proof that $\mathscr{A}/\mathscr{L}$ is an Abelian category, see [3].

The following is immediate.

PROPOSITION 1. *If two groups G and H are isomorphic in the category $\mathscr{A}/\mathscr{F}$, then either $|G| = |H|$ or* $\max(|G|, |H|) < \aleph_0$.

Let G be a reduced p-group, and let α be an ordinal number. Define $p^\alpha G$ inductively as follows: $p^0G = G$, $p^\alpha G = p(p^{\alpha-1}G)$ if $\alpha - 1$ exists, and $p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$ if α is a limit ordinal. The dimension $f_G(\alpha)$ of the vector space $(p^\alpha G)[p]/(p^{\alpha+1}G)[p]$ is called the αth Ulm invariant of G. Define G^α inductively as follows: $G^0 = G$, $G^\alpha = p^\omega(G^{\alpha-1})$ if $\alpha - 1$ exists, and $G^\alpha = \bigcap_{\beta<\alpha} G^\beta$ if α is a limit ordinal. Since G is reduced, there is a least ordinal τ, $|\tau| \leqq |G|$, such that $G^\tau = 0$. The quotient groups $G^\alpha/G^{\alpha+1}$, $\alpha < \tau$, are called the Ulm factors of G. All the Ulm factors except possibly the last, if it exists, are unbounded. An element $x \in G$ has height α in G if $x \in p^\alpha G$, but $x \notin p^{\alpha+1}G$. Ulm's theorem states that two countable reduced p-groups are isomorphic if and only if their corresponding Ulm factors are isomorphic, or equivalently, if and only if they have the same Ulm invariants. This result has been extended to direct sums of countable reduced p-groups by Kolettis in [4]. The author has shown in [1] that two direct sums of countable reduced p-groups G and H are isomorphic in the quotient category $\mathscr{A}/\mathscr{B}$, where $\mathscr{B}$ is the Serre class of bounded groups, if and only if there is an integer $k \geqq 0$ such that for all ordinal numbers α and integers $r \geqq 0$

$$\sum_{j=0}^{r} f_G(\alpha + k + j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha + j)$$

and

$$\sum_{j=0}^{r} f_H(\alpha + k + j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha + j) .$$

Since the Serre class of finite groups is a subclass of the Serre class of bounded groups, it is natural to ask how the above conditions must be strengthened to characterize isomorphism in $\mathscr{A}/\mathscr{F}$. That is the intention of this paper. Any unexplained notation or terminology will be that of Fuchs in [2] with the exception that $G + H$ will denote the sum of G and H while $G \oplus H$ will be the direct sum.

LEMMA 2. *Let S be a cofinite subgroup of a reduced p-group G. Then there is an integer $n \geqq 0$ such that $f_G(\alpha) = f_S(\alpha)$ for all ordinal numbers $\alpha \geqq n$.*

Proof. Since S is cofinite in G, write $G/S = \{x_1 + S, \cdots, x_r + S\}$. Let $n \geqq 0$ be an integer such that $p^n x_i = 0$ for $i = 1, \cdots, r$. Let $p^n y \in p^n G$. Then $y + S = x_i + S$ for some i among $1, \cdots, r$. Thus $y - x_i \in S$ and $p^n y = p^n(y - x_i) \in p^n S$. Hence $p^n G = p^n S$ and $f_G(\alpha) = f_S(\alpha)$ for all ordinal numbers $\alpha \geqq n$.

LEMMA 3. *Let S be a cofinite subgroup of a reduced p-group G. Then for any integer $n \geqq 0$, $f_G(n) = f_S(n)$ or* $\max(f_G(n), f_S(n)) < \aleph_0$.

Proof. Let $f:(p^n S)[p]/(p^{n+1}S)[p] \rightarrow (p^n G)[p]/(p^{n+1}G)[p]$ be the homomorphism induced by the natural injection $S \rightarrow G$. Then $\operatorname{Ker} f = ((p^n S)[p] \cap (p^{n+1}G)[p])/(p^{n+1}S)[p]$ and $|\operatorname{Ker} f| \leqq |(p^{n+1}G)[p]/(p^{n+1}S)[p]| \leqq |(p^{n+1}G/p^{n+1}S)[p]| \leqq |p^{n+1}G/p^{n+1}S| \leqq |G/S|$. Also $\operatorname{Im} f = ((p^n S)[p] + (p^{n+1}G)[p])/(p^{n+1}G)[p]$ and $|\operatorname{Coker} f| = |(p^n G)[p]/((p^n S)[p] + (p^{n+1}G)[p])| \leqq |(p^n G)[p]/(p^n S)[p]| \leqq |G/S|$. Thus $(p^n S)[p]/(p^{n+1}S)[p]$ and $(p^n G)[p]/(p^{n+1}G)[p]$ are isomorphic in $\mathscr{A}/\mathscr{F}$. The lemma now follows from Proposition 1.

LEMMA 4. *Let β be an ordinal number, and let A be a subgroup* *of a reduced p-group G such that A contains no elements of height β or $\beta+1$, the heights taken in G. Then $f_{G/A}(\beta)=f_G(\beta)$. In particular if A is a finite subgroup of G, then $f_{G/A}(\alpha)=f_G(\alpha)$ for almost all ordinal numbers α.*

Proof. Let $\{x_i + A + ((p^{\beta+1}G + A)/A)[p]\}_I$ be a basis of

$$((p^\beta G + A)/A)[p]/((p^{\beta+1}G + A)/A)[p]$$

with $x_i \in p^\beta G$ for each $i \in I$. Then $px_i \in p^{\beta+1}G \cap A$, and since A has no elements of height $\beta+1$ in G, $px_i \in p^{\beta+2}G$. Write $px_i = pz_i$ where $z_i \in p^{\beta+1}G$. Then $x_i - z_i \in (p^\beta G)[p]$, but $x_i - z_i \notin (p^{\beta+1}G)[p]$. Also $x_i - z_i + A + ((p^{\beta+1}G + A)/A)[p] = x_i + A + ((p^{\beta+1}G + A)/A)[p]$. Let $y_i = x_i - z_i$ for $i \in I$. Then $\{y_i + A + ((p^{\beta+1}G + A)/A)[p]\}_I$ is a basis of

$$((p^\beta G + A)/A)[p]/((p^{\beta+1}G + A)/A)[p]$$

with $y_i \in (p^\beta G)[p]$ for each $i \in I$. Suppose $\sum_{i\in J} r_i y_i \in (p^{\beta+1}G)[p]$ for some finite subset $J \subseteqq I$. Then $\sum_{i\in J} r_i y_i + A \in ((p^{\beta+1}G + A)/A)[p]$. Thus $p \mid r_i$ for each $i \in J$ and $\{y_i + (p^{\beta+1}G)[p]\}_I$ is an independent subset of $(p^\beta G)[p]/(p^{\beta+1}G)[p]$. Hence $f_{G/A}(\beta) \leqq f_G(\beta)$.

Let $\{x_i + (p^{\beta+1}G)[p]\}_K$ be a basis of $(p^\beta G)[p]/(p^{\beta+1}G)[p]$. Suppose $\sum_{i\in L} s_i x_i + A \in ((p^{\beta+1}G + A)/A)[p]$ for some finite subset $L \subseteqq K$. Then $\sum_{i\in L} s_i x_i + A = px + A$ for some $x \in p^\beta G$. Now $y = \sum_{i\in L} s_i x_i - px \in p^\beta G \cap A$. Since A has no elements of height β in G, $y \in p^{\beta+1}G$. So

$$\sum_{i\in L} s_i x_i \in (p^{\beta+1}G)[p]\ ,$$

and $p \mid s_i$ for each $i \in L$. Thus $\{x_i + A + ((p^{\beta+1}G + A)/A)[p]\}_K$ is an independent subset of $((p^\beta G + A)/A)[p]/((p^{\beta+1}G + A)/A)[p]$. Hence $f_G(\beta) \leqq f_{G/A}(\beta)$.

LEMMA 5. *Let G be a reduced p-group with finite subgroup A. Then for any ordinal number α, $f_G(\alpha) = f_{G/A}(\alpha)$ or $\max(f_G(\alpha), f_{G/A}(\alpha)) < \aleph_0$.*

Proof. The projection $G \to G/A$ induces a homomorphism

$$f: (p^\alpha G)[p]/(p^{\alpha+1}G)[p] \longrightarrow ((p^\alpha G + A)/A)[p]/((p^{\alpha+1}G + A)/A)[p]\ .$$

The kernel of f is $((p^\alpha G)[p] \cap (p^{\alpha+1}G + A)[p])/(p^{\alpha+1}G)[p]$ and therefore

$$\begin{aligned}|\operatorname{Ker} f| &\leqq |(p^{\alpha+1}G + A)[p]/(p^{\alpha+1}G[p]| \leqq |((p^{\alpha+1}G + A)/p^{\alpha+1}G)[p]| \\ &\leqq |(p^{\alpha+1}G + A)/p^{\alpha+1}G| = |A/p^{\alpha+1}G \cap A| \leqq |A|\ .\end{aligned}$$

Also $\operatorname{Im} f = (((p^\alpha G)[p] + p^{\alpha+1}G + A)/A)[p]/((p^{\alpha+1}G + A)/A)[p]$, Coker f

$\simeq ((p^\alpha G + A)/A)[p]/(((p^\alpha G)[p] + p^{\alpha+1}G + A)/A)[p]$, and $|\mathrm{Coker} f| \leqq |((p^\alpha G + A)/A)[p]/(((p^\alpha G)[p] + A)/A)[p]|$. Let $x + A$, $y + A \in ((p^\alpha G + A)/A)[p]$ with $x, y \in p^\alpha G$. Then $px, py \in A$. If $px = py$, then $x - y \in (p^\alpha G)[p]$, $x - y + A \in (((p^\alpha G)[p] + A)/A)[p]$, and $x + A + (((p^\alpha G)[p] + A)/A)[p] = y + A + (((p^\alpha G)[p] + A)/A)[p]$. So $|\mathrm{Coker} f| \leqq |((p^\alpha G + A)/A)[p]/(((p^\alpha G)[p] + A)/A)[p]| \leqq |A|$. Hence $(p^\alpha G)[p]/(p^{\alpha+1}G)[p]$ and $((p^\alpha G + A)/A)[p]/((p^{\alpha+1}G + A)/A)[p]$ are isomorphic in $\mathscr{A}/\mathscr{F}$. The lemma now follows from Proposition 1.

LEMMA 6. *Let β be an ordinal number, and let G and H be countable reduced p-groups such that*

(i) $f_G(\alpha) = f_H(\alpha)$ *for all ordinal number* $\alpha \neq \beta$, *and*

(ii) $\max(f_G(\beta), f_H(\beta)) < \aleph_0$.

Then G and H are isomorphic in $\mathscr{A}/\mathscr{F}$.

Proof. Take $f_G(\beta) \geqq f_H(\beta)$, and let $k \geqq 0$ be an integer such that $f_G(\beta) = f_H(\beta) + k$. Write $\beta = \omega\gamma + n$, $n \geqq 0$. Then $G_\gamma \simeq H_\gamma \oplus \sum_k C(p^{n+1})$ where $G_\gamma = G^\gamma/G^{\gamma+1}$ and $H_\gamma = H^\gamma/H^{\gamma+1}$. Suppose H_γ is finite. Then so is G_γ. Moreover, $G/G^\gamma \simeq H/H^\gamma$, and so G and H are isomorphic in $\mathscr{A}/\mathscr{F}$. Assume H_γ is infinite. For each $\alpha < \gamma$, write $H_\alpha = L_\alpha \oplus M_\alpha$ where L_α and M_α are unbounded and $|L_\alpha| = |M_\alpha| = |H_\alpha|$. Let L be a countable reduced p-group whose Ulm factors are H_α for $\alpha \geqq \gamma$ and L_α for $\alpha < \gamma$. (Such a group exists by Zippin's lemma.) Let M be a countable reduced p-group whose Ulm factors are M_α for $\alpha < \gamma$ and $M_\gamma = \sum_k C(p^{n+1})$. Then $L \oplus M \simeq G$ and $L \oplus (M/M^\gamma) \simeq H$. Hence G and H are isomorphic in $\mathscr{A}/\mathscr{F}$.

LEMMA 7. *Let β be an ordinal number, and let G and H be direct sums of countable reduced p-groups such that*

(i) $f_G(\alpha) = f_H(\alpha)$ *for all ordinal numbers* $\alpha \neq \beta$, *and*

(ii) $\max(f_G(\beta), f_H(\beta)) < \aleph_0$.

Then G and H are isomorphic in $\mathscr{A}/\mathscr{F}$.

Proof. Let $G = \sum'_{\lambda \in \Lambda} X_\lambda$ where $|X_\lambda| \leqq \aleph_0$ for each $\lambda \in \Lambda$. Let $\Lambda_1 = \{\lambda \in \Lambda \mid f_{X_\lambda}(\beta) = 0\}$, and let $X = \sum_{\lambda \in \Lambda - \Lambda_1} X_\lambda$. Since $f_G(\beta)$ is finite, $|\Lambda - \Lambda_1| < \aleph_0$. There is a countable reduced p-group Y with $f_Y(\alpha) = f_X(\alpha)$ for $\alpha \neq \beta$ and $f_Y(\beta) = f_H(\beta)$. By Lemma 6, X and Y are isomorphic in $\mathscr{A}/\mathscr{F}$. Hence $\sum_{\lambda \in \Lambda_1} X_\lambda \oplus X$ and $\sum_{\lambda \in \Lambda_1} X_\lambda \oplus Y$ are isomorphic in $\mathscr{A}/\mathscr{F}$. But $G \simeq \sum_{\lambda \in \Lambda_1} X_\lambda \oplus X$ and $H \simeq \sum_{\lambda \in \Lambda_1} X_\lambda \oplus Y$. Hence G and H are isomorphic in $\mathscr{A}/\mathscr{F}$.

THEOREM. *Let G and H be direct sums of countable reduced p-groups. Then G and H are isomorphic in $\mathscr{A}/\mathscr{F}$ if and only if*

(i) $f_G(\alpha) = f_H(\alpha)$ *for almost all ordinal numbers* α,

and

(ii) $f_G(\alpha) \neq f_H(\alpha)$ *implies* $\max(f_G(\alpha), f_H(\alpha)) < \aleph_0$.

Proof. That (i) and (ii) are necessary follows from Lemmas 2, 3, 4, and 5. That (i) and (ii) are sufficient follows from a finite number of applications of Lemma 7.

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Received January 8, 1968.

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