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THE LATTICE OF PRETOPOLOGIES ON AN ARBITRARY SET S

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The structure of the lattice of pretopologies on the set S, unlike that of the lattice of topologies on S (a proper sublattice of the former), has not been closely examined. We establish that pretopologies may be identified with products of certain filters in a natural way. From this identification, we are able to determine much of the structure of this lattice.

We show that $(p(S), \leq)$, the lattice of pretopologies (pretopologies in the sense of Kent [2; p. 126]) on the set S, is order isomorphic to a sublattice of filters on S^S (using Bourbaki's [1; p. 61–63] approach to filters). From this, we deduce that $(p(S), \leq)$ is complete, atomic, coatomic, modular, distributive, and compactly generated; S being finite is both necessary and sufficient for the lattice to be co-compactly generated and complemented (in which case it has a unique complement). It is infinitely distributive only in the trivial case of S being finite. (The lattice terminology is that of Szász [3] with the exception of coatomic which we use rather than dually atomic and co-compactly generated which is used for the notion dual to that of compactly generated.)

1. The isomorphism φ. A pretopology p on a set S is completely determined by a specification of the neighborhood filter $\eta_p(x)$ of each x in S. These neighborhoods necessarily satisfy $\eta_p(x) \leq \overline{x}$, where $\overline{x}$ is the principal filter generated by $\{x\}$. For each $x \in S$, let $\underline{F}(x) = \{\mathfrak{F}: \mathfrak{F} \geq \overline{x}, \mathfrak{F} \text{ a filter on } S\}$, and let $\underline{F} = \prod_{x\in S} \underline{F}(x)$ (Bourbaki [1; p. 69–70]); both ordered by $\mathfrak{F} \leq \mathfrak{G}$ if and only if $\mathfrak{F} \subset \mathfrak{G}$. Then $\underline{F}$ is a subset of the set of filters on S^S. Indeed, it is easily seen that $\underline{F}$, with this ordering, is a sublattice of the lattice of filters on S^S. For given $\mathfrak{F}, \mathfrak{G} \in \underline{F}$, we have $\mathfrak{F} \wedge \mathfrak{G} = \{F \cup G: F \in \mathfrak{F}, G \in \mathfrak{G}\}$ and $\mathfrak{F} \vee \mathfrak{G} = \{F \cap G: F \in \mathfrak{F}, G \in \mathfrak{G}\}$ ($F \cap G \neq \phi$ since $\prod_{x\in S}\{x\} \in F, G$).

Given a pretopology p, we define φ by $\varphi(p) = \prod_{x\in S} \eta_p(x)$. Then φ is easily seen to be a one-to-one mapping from the pretopologies on S *onto* $\underline{F}$. Furthermore, if p, q are pretopologies on S, the following will be equivalent:

(1) $p \leq q$;

(2) $\eta_p(x) \leq \eta_q(x)$ for all x in S; and

(3) $\prod_{x\in S} \eta_p(x) \leq \prod_{x\in S} \eta_q(x)$.

Thus we have

THEOREM 1. *φ is an order isomorphism from the lattice of pretopologies on S onto the sublattice $(\underline{F}, \leq)$ of filters on S^S.*

2. **The structure of $(\underline{F}(x), \leq)$.** We shall deduce the structure of $(\underline{F}, \leq)$ from an examination of the structures of the lattices $(\underline{F}(x), \leq)$, for each x.

It follows readily from the definition of $\wedge$ and $\vee$ in $(\underline{F}(x), \leq)$ that this lattice is complete, modular, and distributive. The remaining propositions of this section further describe its structure.

PROPOSITION 1. $(\underline{F}(x), \leq)$ is atomic. Its atoms are precisely those elements of the form $\overline{S\setminus\{a\}}$ for $a \neq x$. ($\overline{A}$ denotes the filter of all super-sets of A in S).

Proof. Given $\mathfrak{F} \neq 0 \equiv \overline{S}$ in $\underline{F}(x)$, select $A \in \mathfrak{F}$, $A \neq S$. Then there exists an $a \neq x$ in $S\setminus A$, and $\overline{S\setminus\{a\}} \leq \mathfrak{F}$.

To show that $\overline{S\setminus\{a\}}$ is an atom of $(\underline{F}(x), \leq)$ for $a \neq x$, let $\mathfrak{F} < \overline{S\setminus\{a\}}$. Then $S\setminus\{a\} \subset F$ for all $F \in \mathfrak{F}$, and $F \subset S\setminus\{a\}$ for no $F \in \mathfrak{F}$. Thus $\mathfrak{F} = \overline{S} = 0$.

PROPOSITION 2. $(\underline{F}(x), \leqq)$ is coatomic. Its coatoms are precisely those $\mathfrak{F} = \overline{x} \wedge \mathfrak{U}$ where $\mathfrak{U} \neq \overline{x}$ is an ultrafilter.

Proof. Let $\mathfrak{F} \in \underline{F}(x)$ be distinct from $1 \equiv \overline{x}$. Then since $\mathfrak{F}$ is not an ultrafilter, there must be at least two ultrafilters above $\mathfrak{F}$. One of these, say $\mathfrak{U}$, must be distinct from $\overline{x}$. Then $\mathfrak{F} \leq \overline{x} \wedge \mathfrak{U}$.

To show that $\overline{x} \wedge \mathfrak{U}$ is a coatom of $(\underline{F}(x), \leq)$, assume there is an $\mathfrak{F} \in \underline{F}(x)$ with $\overline{x} \wedge \mathfrak{U} < \mathfrak{F} < \overline{x}$. Since $\mathfrak{F} < \overline{x}$, $\{F\setminus\{x\}: F \in \mathfrak{F}\}$ is a base for some filter $\mathfrak{G}$. Clearly $\overline{x} \wedge \mathfrak{G} = \mathfrak{F}$. Now, for each $U \in \mathfrak{U}$, there exists $F \in \mathfrak{F}$ such that $F \subset U \subset \{x\}$, since $\overline{x} \wedge \mathfrak{U} < \mathfrak{F}$. Thus $F\setminus\{x\} \subset U$. Hence $\mathfrak{G} \geq \mathfrak{U}$. But $\mathfrak{U}$ is an ultrafilter. Consequently $\mathfrak{G} = \mathfrak{U}$. Thus, we must conclude that $\overline{x} \wedge \mathfrak{U} = \overline{x} \wedge \mathfrak{G} = \mathfrak{F}$, a contradiction.

PROPOSITION 3. $\mathfrak{F} \in \underline{F}(x)$ is compact if and only if $\mathfrak{F} = \overline{A}$ for some $A \subset S$ with $x \in A$. Consequently $(\underline{F}(x), \leq)$ is compactly generated.

Proof. Let $\mathfrak{F} \in \underline{F}(x)$ be compact. Observe that $\mathfrak{F} = \bigvee \{\overline{F}: F \in \mathfrak{F}\}$. Thus $\mathfrak{F} \leq \bigvee_{i=1}^{n} \overline{F}_i$ for some choice of n and $F_i \in \mathfrak{F}$ $(i = 1, \cdots n)$. But since filters include finite intersections of their members, $\mathfrak{F} \geqq \overline{\bigcap_{i=1}^{n} F_i} \equiv \bigvee_{i=1}^{n} \overline{F}_i$. Thus $\mathfrak{F} \equiv \overline{\bigcap_{i=1}^{n} F_i}$.

Conversely, let $\mathfrak{F} = \overline{A}$ and let $\mathfrak{F} \leq \bigvee_{\gamma \in \Gamma} \mathfrak{F}_\gamma$. Then since $A \in \mathfrak{F}$, there exists $\Gamma_0 \subset \Gamma$ (Γ_0 finite), and $F_\gamma \in \mathfrak{F}_\gamma$ such that $\bigcap_{\gamma \in \Gamma} F_\gamma \subset A$.

Thus $\mathfrak{F} \leq \bigvee_{\gamma \in \Gamma_0} F_\Gamma$.

For any $\mathfrak{G} \in \underline{F}(x)$, we have

$$\begin{aligned}\bigvee\{\mathfrak{F}\colon \mathfrak{F} \leq \mathfrak{G}, \mathfrak{F} \text{ compact}\} \leq \mathfrak{G} &= \bigvee\{\overline{G}\colon G \in \mathfrak{G}\} \\ &\leq \bigvee\{\mathfrak{F}\colon \mathfrak{F} \leq \mathfrak{G}, \mathfrak{F} \text{ compact}\}\end{aligned}$$

thus $\mathfrak{G} = \bigvee\{\mathfrak{F}\colon \mathfrak{F} \leq \mathfrak{G}, \mathfrak{F} \text{ compact}\}$ and $(\underline{F}(x), \leq)$ is compactly generated.

PROPOSITION 4. $\mathfrak{F} \in \underline{F}(x)$ is co-compact if and only if $\mathfrak{F} = \overline{A}$ where A is some finite subset of S containing x. Consequently $(\underline{F}(x), \leq)$ is co-compactly generated if and only if S is finite.

Proof. Let $\mathfrak{F} \in \underline{F}(x)$ be co-compact and let $T = S\backslash\{x\}$. Observe that $\mathfrak{F} \geq \overline{S} \equiv \bigwedge_{a \in T} \overline{\{x, a\}}$. Consequently for some n and $a_i \in T\,(i = 1, \cdots, n)$, $\mathfrak{F} \geq \bigwedge_{i=1}^n \overline{\{x, a_i\}} = \overline{\bigcup_{i=1}^n \{x, a_i\}}$. Thus $\bigcup_{i=1}^n \{x, a_i\} \in \mathfrak{F}$. But any filter containing a finite set B can be expressed as $\overline{A}$ for some $A \subseteqq B$. Thus $\mathfrak{F} = \overline{A}$ for some $A \subset \bigcup_{i=1}^n \{x, a_i\}$.

Conversely, let $\mathfrak{F} = \overline{A}$ where $A = \{x, a_1, a_2 \cdots a_n\}$, and suppose that $\mathfrak{F} \geq \bigwedge_{\gamma \in \Gamma} \mathfrak{F}_\gamma$. Then we may select $F_\gamma \in \mathfrak{F}_\gamma$ such that $\bigcup_{\gamma \in \Gamma} F_\gamma \supseteq A$. Select γ_i so that $a_i \in F_{\gamma_i}$. Then $\mathfrak{F} \geq \bigwedge_{i=1}^n \mathfrak{F}_{\gamma_i}$.

If S is finite, each $\mathfrak{F} \in \underline{F}(x)$ is of the form $\overline{A}$ with A finite, so $(\underline{F}(x), \leq)$ will consist only of co-compact elements and hence be co-compactly generated. Observe however, that $\wedge_{\gamma \in \Gamma} \overline{A_\gamma} = \overline{\bigcup_{\gamma \in \Gamma} A_\gamma}$ for arbitrary filters. Thus, in particular, the only elements of $(\underline{F}(x), \leq)$ which will be co-compactly generated are the principal filters. Consequently $(\underline{F}(x), \leq)$ is not co-compactly generated when S is infinite.

PROPOSITION 5. $\mathfrak{F} \in \underline{F}(x)$ has a complement $\mathfrak{G}$ if and only if $\mathfrak{F} = \overline{A}$. In this case $\mathfrak{G}$ is unique and $\mathfrak{G} = \overline{(S\backslash A) \cup \{x\}}$. Consequently $(\underline{F}(x), \leq)$ is complemented if and only if S is finite.

Proof. Let $\mathfrak{F} = \overline{A}$. If $\mathfrak{G} = \overline{(S\backslash A) \cup \{x\}}$, then $\mathfrak{F} \wedge \mathfrak{G} = \overline{S} \equiv 0$ and $\mathfrak{F} \vee \mathfrak{G} = \overline{x} \equiv 1$. Thus $\mathfrak{G}$ is a complement of $\mathfrak{F}$. Let $\mathfrak{G}'$ be any complement of $\mathfrak{F}$. Then since $\mathfrak{F} \wedge \mathfrak{G}' = \overline{S}$, $(S\backslash A) \subset \{x\}$ must be in $\mathfrak{G}'$. But $\mathfrak{F} \vee \mathfrak{G}' = \overline{x}$, so no proper subset of $(\underline{S}\backslash\underline{A}) \cup \{x\}$ may be in $\mathfrak{G}'$. Consequently $\mathfrak{G}' = \overline{(S\backslash A \cup \{x\}} = \mathfrak{G}$.

Suppose on the other hand, that $\mathfrak{F}$ is not principal. Let $A = \bigcap \mathfrak{F}$. Then $A \neq \phi$ since $x \in A$. Suppose that $\mathfrak{G}$ is a complement of $\mathfrak{F}$. Then for each $F \in \mathfrak{F}$, $G \in \mathfrak{G}$, we have $F \cup G = S$, since $\mathfrak{F} \wedge \mathfrak{G} = \overline{S}$. Thus $B = (S\backslash A) \cup \{x\}$ must be a subset of every G in $\mathfrak{G}$. Observe that any F in $\mathfrak{F}$ will contain A as a proper subset since $\mathfrak{F}$ is nonprincipal. Thus any $F \in \mathfrak{F}$ will include points of B distinct from x. Hence for each $G \in \mathfrak{G}$, $F \vee G$ will contain at least two points. But this violates the requirement that $\mathfrak{F} \vee \mathfrak{G} = \overleftarrow{x}$. Therefore $\mathfrak{F}$ can not have a complement.

We conclude this section with a discussion of infinite distributivity. Let $\mathfrak{F}, \mathfrak{F}_\gamma \in \underline{F}(x)$ $(\gamma \in \Gamma)$ be arbitrary. Then, since filter joins are given by finite intersections, we have $\mathfrak{F} \wedge (\bigvee_{\gamma \in \Gamma} \mathfrak{F}_\gamma) = \bigvee_{\gamma \in \Gamma}(\mathfrak{F} \wedge \mathfrak{F}_\gamma)$. We also have $\mathfrak{F} \vee (\bigwedge_{\gamma \in \Gamma} \mathfrak{F}_\gamma) \leqq \bigwedge_{\gamma \in \Gamma}(\mathfrak{F} \vee \mathfrak{F}_\gamma)$. However, if $\underline{S}$ is not finite, we need not have equality. A particular example can be found by letting $\Gamma = S$, $\mathfrak{F}_\gamma = \overline{\{\gamma, x\}}$, and $\mathfrak{F} = \{A: x \in A, A \text{ cofinite}\}$. For in this case $\{s\} \in \bigwedge_{\gamma \in \Gamma}(\mathfrak{F} \vee \mathfrak{F}_\gamma)$ is not cofinite. Thus $(\underline{F}(x), \leq)$ is distributive only in the trivial case where S, and consequently $\underline{F}(x)$, is finite.

3. **Structure of $(\underline{F}, \leq)$ and $(\underline{p}(S), \leq)$.** The results of § 2 carry directly over to the lattice $(\underline{F}, \leq)$. For letting $\mathfrak{F} = \prod_{x \in S} \mathfrak{F}_x$, $\mathfrak{G} = \prod_{x \in S} \mathfrak{G}_x$ with $\mathfrak{F}_x, \mathfrak{G}_x \in \underline{F}(x)$ for each x, we see that $\mathfrak{F} \leq \mathfrak{G}$ if and only if $\mathfrak{F}_x \leq \mathfrak{G}_x$ in $(\underline{F}(x), \leq)$ for each x, while $\mathfrak{F} \wedge \mathfrak{G} = \prod_{x \in S}(\mathfrak{F}_x \wedge \mathfrak{G}_x)$ and $\mathfrak{F} \vee \mathfrak{G} = \prod_{x \in S}(\mathfrak{F}_x \vee \mathfrak{G}_x)$. We summarize these results in the following proposition. Each of its components follows from the coresponding result in § 2.

Proposition 6. 1. $(\underline{F}, \leq)$ is complete, modular, and distributive. It is infinitely distributive only in the trivial case of S, and consequently $\underline{F}$, being finite.

2. $(\underline{F}, \leq)$ is atomic (coatomic). $\mathfrak{F} = \prod_{x \in S} \mathfrak{F}_x \in \underline{F}$ is an atom (coatom) if and only if $\mathfrak{F}_x = \overline{S}$ for $x \neq s$ ($\mathfrak{F}_x = \overleftarrow{x}$ for $x \neq s$) and $\mathfrak{F}_s$ is an atom of $\underline{F}(s)$ (a coatom of $(\underline{F}(s))$.

3. $\mathfrak{F} \in \underline{F}$ is compact (co-compact) if and only if $\mathfrak{F}_x$ is compact (co-compact) for each $x \in \overline{S}$ and $\mathfrak{F}_x = \overline{S}$ ($\mathfrak{F}_x = \overleftarrow{x}$) except for most a finite number of the $x \in S$.

4. $(\underline{F}, \leq)$ is compactly generated.

5. $(\underline{F}, \leq)$ is co-compactly generated if and only if S is finite.

6. $\mathfrak{F}$ has a complement $\mathfrak{G} = \prod_{x \in S} \mathfrak{G}_x$ if and only if $\mathfrak{F}_x$ and $\mathfrak{G}_x$ are complements for each $x \in S$.

7. $(\underline{F}, \leq)$ is complemented if and only if S is finite. In this case complements will be unique.

Using the isomorphism φ, these results immediately carry over to $(\underline{p}(S), \leq)$. Thus we have

Theorem 2. *$(\underline{p}(S), \leq)$ is always complete, modular, distributive, atomic, coatomic, and compactly generated. It is complemented (and has unique complements), co-compactly generated, and infinitely distributive if and only if S is finite.*

References

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