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THE POWER-COMMUTATOR STRUCTURE OF FINITE p-GROUPS

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For a finite p-group G, G_n is the n-th element in the descending central series of G; $P(G)$ is the subgroup of G generated by the set of all x^p for x belonging to G; and $\Phi(G)$ is the Frattini subgroup of G.

Hobby has characterized finite p-groups G (for $p > 2$) in which $P(G) = \Phi(G)$. Since $\Phi(G) = G_2P(G)$, the condition $P(G) = \Phi(G)$ is clearly equivalent to $G_2 \subseteqq P(G)$. In this paper we examine the class of finite p-groups G which have the property that $G_n \subseteqq P(G_m)$ for $1 < n/m < p$. In § 2 we consider consequences of this property in the case $m = 1$. For example, if $G_{p-1} \subseteqq P(G)$, then the product of p-th powers of elements of G is the p-th power of an element of G (Theorem 2). In § 3 we examine some connections between the property $G_n \subseteqq P(G_m)$ and regularity, and obtain a characterization of regular 3-groups (Theorem 4). In § 4 we obtain bounds on the number of generators of various commutator subgroups of G in the case $G_3 \subseteqq P(G)$, $p > 3$.

For a discussion of p-groups G for which $G_2 \subseteqq P(G)$ see [6].

1. Notation. Throughout this paper G is a finite p-group. If $X_1, X_2, \cdots, X_n$ are subsets of G, then $\langle X_1, X_2, \cdots, X_n \rangle$ is the smallest subgroup of G containing all the X_i. If $X = \{x\}$ for some element x, we write $X = x$. We denote by $d(G)$ the minimal number of elements of G which generate G, while $|G|$ is the order of G. We set $P^n(G) = \langle \{x^{p^n} \mid x \in G\} \rangle$. Also, $Z(G)$ is the center of G and $\Phi(G)$ is the Frattini subgroup of G.

Simple commutators of weight n are defined inductively by setting $(x_1, x_2) = x_1^{-1}x_2^{-1}x_1x_2$ and $(x_1, \cdots, x_n) = ((x_1, \cdots, x_{n-1}), x_n)$ for $n > 2$. In addition, we define $(x, 1y) = (x, y)$ and $(x, ny) = (x, (n-1)y, y)$ for $n > 1$. For subgroups $H_1, H_2, \cdots, H_n$ of G we set

$$(H_1, H_2, \cdots, H_n) = \langle \{(h_1, h_2, \cdots, h_n) \mid h_i \in H_i\} \rangle .$$

Similarly, $(H_1, 1H_2) = (H_1, H_2)$ and $(H_1, nH_2) = (H_1, (n-1)H_2, H_2)$ for $n > 1$. The descending central series of G is defined by setting $G_1 = G$ and $G_n = (G_{n-1}, G)$ for $n > 1$. A group G is said to have *class c* if $G_{c+1} = 1$ and $G_c \neq 1$. Finally, the derived series of G is defined by setting $G^{(0)} = G$ and $G^{(i+1)} = (G^{(i)}, G^{(i)})$ for $i \geqq 0$.

2. Basic results. It is known ([4], Th. 3.1, p. 63) that when-

ever x and y belong to G,

$$(*) \qquad (xy)^p = x^p y^p cd$$

where $c \in P(\langle x, y\rangle_2)$ and $d \in \langle x, y\rangle_p$. Applying this result to the expression $(a^p, b) = a^{-p}(a(a, b))^p$ one can obtain the following lemma by repeated induction.

LEMMA 1. *If* $s, n, k \geqq 1$, *then* $(P(G_n), sG_k) \subseteqq P(G_{n+sk})G_{pn+sk}$.

THEOREM 1. *Let n and m be integers and p be a prime such that* $1 < n/m < p$. *If* $G_n \subseteqq P(G_m)$, *then* $G_{n+k} \subseteqq P(G_{m+k})$ *for* $k \geqq 0$.

Proof. We proceed by induction on k, the case $k = 0$ being the hypothesis. Suppose that $G_{n+k} \subseteqq P(G_{m+k})$ and that G is a group of minimal order for which $G_{n+k+1} \not\subseteqq P(G_{m+k+1})$. Clearly we may assume $P(G_{m+k+1}) = 1$. It follows from Lemma 1 that $(P(G_{m+k}), G) \subseteqq G_{p(m+k)+1}$. Hence $G_{n+k+1} \subseteqq (P(G_{m+k}), G) \subseteqq G_{p(m+k)+1}$. However, $p(m + k) + 1 > n + k + 1$, so $G_{n+k+1} \subset G_{n+k+1}$, a contradiction. Thus $G_{n+k+1} \subseteqq P(G_{m+k+1})$.

REMARK. We shall be most concerned with the case $m = 1$ of Theorem 1: If $G_n \subseteqq P(G)$ and $n < p$, then $G_{n+k} \subseteqq P(G_{1+k})$ for $k \geqq 0$. In Example 1 we show that this result cannot be extended to the case $n \geqq p$.

COROLLARY 1.1. *If $n < p$ and* $G_n \subseteqq P(G)$, *then*
(a) $(G_i)_n \subseteqq P(G_i)$ *for* $i = 1, 2, 3, \cdots$,
(b) $(P(G))_n \subseteqq P(G_n) \subseteqq P(P(G))$, *and*
(c) *for any* $x \in G$, *if* $H = \langle G_2, x\rangle$, *then* $H_n \subseteqq P(G_2) \subseteqq P(H)$.

Proof. (a) It is known ([4], Th. 2.55, p. 55) that $(G_i)_n \subseteqq G_{in}$. Since $in - (n - 1) \geqq i$ it follows from Theorem 1 that

$$G_{in} \subseteqq P(G_{in-(n-1)}) \subseteqq P(G_i) .$$

(b) It follows from Lemma 1 that $(P(G))_n \subseteqq (P(G), (n - 1)G) \subseteqq P(G_n)G_{p+n-1}$. By Theorem 1, $G_{p+n-1} \subseteqq P(G_p)$, so

$$(P(G))_n \subseteqq P(G_n)P(G_p) \subseteqq P(P(G)) .$$

(c) Since G_2 is central modulo G_3 and H/G_2 is cyclic, we have $H_2 \subseteqq G_3$. It follows that $H_i \subseteqq G_{i+1}$ for $i \geqq 2$. By Theorem 1, $G_{n+1} \subseteqq P(G_2)$. Thus $H_n \subseteqq G_{n+1} \subseteqq P(G_2) \subseteqq P(H)$.

COROLLARY 1.2. *If* $n < p$, $G_n \subseteqq P(G)$, *and t is an integer such that* $2^t \geqq n + 1$, *then* $G^{(k+t-1)} \subseteqq P(G^{(k)})$ *for* $k \geqq 1$.

Proof. We assume that the result holds for all groups of order less than $|G|$. It follows from Corollary 1.1 that $G^{(1)}$ satisfies the hypothesis of this corollary. Since $|G^{(1)}| < |G|$ we have

$$(**) \qquad (G^{(1)})^{(k+t-1)} \subseteqq P((G^{(1)})^{(k)})$$

for $k \geqq 1$.

By Theorem 2.54 of [4], $G^{(t)} \subseteqq G_{2t}$. Hence for $k = 1$ it follows from Theorem 1 that $G^{(t)} \subseteqq G_{n+1} \subseteqq P(G_2) = P(G^{(1)})$. If $k > 1$ we replace k by $k - 1$ in (**) and obtain

$$G^{(k+t-1)} = (G^{(1)})^{(k-1+t-1)} \subseteqq P((G^{(1)})^{(k-1)}) = P(G^{(k)}) .$$

REMARK. When $n = t = 2$ in Corollary 1.2 we obtain Theorem 2 of [6].

We now show that Theorem 1 for the case $m = 1$ cannot be extended to include $n \geqq p$.

EXAMPLE 1. Let $\langle a \rangle \wr \langle b \rangle$ be the wreath product of $\langle a \rangle$ by $\langle b \rangle$, where $a^p = b^{p^r} = 1$ and $r > 0$. Then $G_p \subseteqq P(G)$, $P(G_2) = 1$, and $G_{p^r} \neq 1$.

It is clear that the property $G_n \subseteqq P(G)$, $n < p$, is inherited by factor groups and preserved by direct products. By the following example we show that this property is not always inherited by a subgroup H of G.

EXAMPLE 2. Let $W = \langle a \rangle \wr \langle b \rangle$, where $a^p = b^p = 1$. For $2 \leqq n \leqq p - 1$, set $H = W/W_{n+1}$ and $H_n = \langle z \rangle$. Let $\langle d \rangle$ be the cyclic group of order p^2, and G be the group formed by taking the direct product of H and $\langle d \rangle$ with the amalgamation $d^p = z$. Then $G_n = H_n = \langle z \rangle = P(G)$, while $P(H) = 1$.

THEOREM 2. *If $G_n \subseteqq P(G)$ and $n < p$, then for any $x_1, \cdots, x_k$ in G, there is an element h in G such that $x_1^p \cdots x_k^p = h^p$.*

Proof. The result is clear if G is abelian. Suppose that G is nonabelian and that the theorem holds for all groups H with $|H| < |G|$. It follows from (*) that $(x_1 \cdots x_k)^p = x_1^p \cdots x_k^p g_1^p \cdots g_t^p g$, where $g_i \in G_2$ for $1 \leqq i \leqq t$ and $g \in G_p$. By Theorem 1, $G_p \subseteqq P(G_2)$, so there exist elements $g_{t+1}, \cdots, g_r$ in G_2 such that $g = g_{t+1}^p \cdots g_r^p$.

By Corollary 1.1, $(G_2)_n \subseteqq P(G_2)$. Since $|G_2| < |G|$ it follows from the induction hypothesis applied to G_2 that $g_1^p \cdots g_t^p g_{t+1}^p \cdots g_r^p = y^p$, where $y \in G_2$. That is, $x_1^p \cdots x_k^p = (x_1 \cdots x_k)^p s^p$, where $s = y^{-1}$ is in

G_2. Next set $x = x_1 \cdots x_k$ and let $H = \langle G_2, x\rangle$. By Corollary 1.1, $H_n \subseteqq P(H)$. It follows from the Burnside Basis Theorem (see e.g. [3], p. 176) that $d(G) = d(G/K)$ if K is a normal subgroup of G and $K \subseteqq \Phi(G)$. Thus, since G is nonabelian, $H \subset G$. Hence, applying the induction hypothesis to H, $x^p s^p = h^p$ for some h in H. Therefore $x_1^p \cdots x_k^p = h^p$.

COROLLARY 2.1. *If* $G_n \subseteqq P(G)$ *and* $n < p$, *then* $P(P(G)) = P^2(G)$.

REMARK. The results of Theorem 2 and Corollary 2.1 are the best possible. That is, if $n \geqq p$ then it does not follow from $G_n \subseteqq P(G)$ that the products of p-th powers are p-th powers or that $P(P(G)) = P^2(G)$. For if we let $G = \langle a\rangle \wr \langle b\rangle$, where $a^{p^2} = b^{p^2} = 1$, then it can be shown that $G_p \subseteqq P(G)$, while $b^{-p}(ba_0)^p$ is not a p-th power for some $a_0, b \in G$, and $P^2(G) \neq P(P(G))$.

3. Regularity. A p-group G is *regular* if for each pair of elements a, b of G, $(ab)^p = a^p b^p c$ where $c \in P(\langle a, b\rangle_2)$. If G is not regular, G is called *irregular*. It follows from (*) that G is regular if $\langle a, b\rangle_p \subseteqq P(\langle a, b\rangle_2)$ for each 2-generator subgroup $\langle a, b\rangle$ of G. By comparison, $G_p \subseteqq P(G_2)$ whenever $G_n \subseteqq P(G)$ and $n < p$. In addition, the result of Theorem 2 is also true in regular p-groups. Thus the property $G_n \subseteqq P(G)$, $n < p$, is similar to regularity. However, neither of these properties implies the other, as is shown in the next two examples.

First we construct a regular group G for which $G_{p-1} \not\subseteq P(G)$.

EXAMPLE 3. Let $W = \langle a\rangle \wr \langle b\rangle$, where $a^p = b^p = 1$. Set $G = W/W_p$. Since $W_p = P(W)$, clearly $G_{p-1} \neq 1$ and $P(G) = 1$. However, G has class $p-1$, and is thus regular ([4], Corollary 4.13, p. 73).

Next we construct an irregular group G for which $G_2 \subseteqq P(G)$.

EXAMPLE 4. Let $H = \langle a, b\rangle$, where $a^{p^p} = b^{p^{p-1}} = 1$ and $b^{-1}ab = a^{p+1}$. Then $(a, nb) = a^{p^n}$, so $H_2 \subseteqq \langle a^p\rangle$. Thus $|H_2| = p^{p-1}$ and $H_{p+1} = 1$. On the other hand, $(a, (p-1)b) \neq 1$, so $H_p \neq 1$. Thus H has class p, H_2 is abelian and $d(H) = 2$. It follows from Theorem 1.4 of [7] that there is a positive integer n such that if $H_i = H (i = 1, \cdots, n)$, then $G = H_1 \times \cdots \times H_n$ is irregular. However, it is clear that $G_2 \subseteqq P(G)$.

We know from Example 4 that $G_2 \subseteqq P(G)$ does not imply regularity. However, in that example $d(G) > 2$. We now show that in a finite 2-generator p-group $(p \neq 2)$ $G_2 \subseteqq P(G)$ does imply regularity.

THEOREM 3. *Let* G *be a finite* p-*group* $(p \neq 2)$ *with* $G_2 \subseteqq P(G)$

and $d(G) = 2$. *Then* G *is regular.*

Proof. By Theorem 1, $G_3 \subseteqq P(G_2)$. Hence $d(G_2/P(G_2)) \leqq d(G_2/G_3)$. It follows from Theorem 2.83 of [4] that $d(G_2/G_3) \leqq 1$. By Corollary 1.1, $(G_2)_2 \subseteqq P(G_2)$, so $G_2/P(G_2)$ is an elementary abelian p-group. Thus $[G_2: P(G_2)] \leqq p$, and G is regular by Theorem 2.3 of [5].

We next obtain a characterization of regular 3-groups.

THEOREM 4. *If G is a finite 3-group, then G is regular if, and only if, $H_3 \subseteqq P(H_2)$ for each 2-generator subgroup H of G.*

Proof. It follows from (*) that the latter condition implies regularity. On the other hand, if G is regular, then all subgroups of G are regular. Alperin ([1], Lemma 3.1.1, p. 96) has shown that if H is a regular 2-generator 3-group, then its derived group is cyclic. Hence $H_3 \subseteqq P(H_2)$.

REMARK. If $p = 3$ or $p = 2$ and G is a regular 2-generator p-group, then $G_p \subseteqq P(G_2)$. However, these are the only primes for which this result holds, since the Burnside group of exponent p and 2 generators has class greater than p when $p > 3$.

As in the proof of Theorem 4, if G_i is cyclic, then $G_{i+1} \subseteqq P(G_i)$. In particular, $G_3 \subseteqq P(G_2)$ if $d(G_2) = 1$. If $d(G_2) = 2$ a theorem of Blackburn gives a similar result.

THEOREM 5. *Let G be a finite p-group such that $d(G_2) = 2$. Then $G_4 \subseteqq P(G_2)$.*

Proof. We may assume $P(G_2) = 1$. It follows from Theorem 1 of [2] that $[G_2: P(G_2)] \leqq p^2$, so $G_4 = 1$.

We now show that for each prime p and each integer $n \geqq 3$, there is a finite p-group G such that $d(G_2) = n$ and $G_4 \nsubseteq P(G_2)$. This shows that the result of Theorem 5 is not true if $d(G_2) > 2$.

EXAMPLE 5. Let $W = \langle a \rangle \wr \langle b \rangle$, where $a^p = b^{p^3} = 1$. Then $| W_i / W_{i+1} | = p$ for $i \geqq 2$ and W has class p^3. Thus $W_5 \neq 1$. Let $H = W/W_5$. Then H_2 is an elementary abelian p-group, $d(H_2) = 3$, $H_4 \neq 1$, and $P(H_2) = 1$. Thus $H_4 \nsubseteq P(H_2)$. If $n = 3$ we may let $G = H$. If $n > 3$, let D be one of the nonabelian groups of order p^3. Then $| D_2 | = p$. Let K be the group formed by taking the direct product on $n - 3$ copies of D. Set $G = H \times K$. Then $G_2 = H_2 \times K_2$ and $d(G_2) = d(H_2) + (n - 3) = n$. Clearly $G_4 \nsubseteq P(G_2)$.

4. Bounds on generators of commutator subgroups. Hobby ([6], Th. 3, p. 855) has shown that the condition $G_2 \subseteqq P(G)$ $(p > 2)$ imposes restrictions on the generating elements of $G^{(i)}$ for $i \geqq 0$. In this section we obtain similar results in the case $G_3 \subseteqq P(G)$ and $p > 3$. The procedure used here can be extended to the general case $G_n \subseteqq P(G)$, $n < p$, although the estimates thus obtained are not as precise.

THEOREM 6. *Suppose* $p > 3$, $G_3 \subseteqq P(G)$, *and* $d = d(G)$. *Then* $d(G_3) \leqq (1/2)d(d^2 - 1)$.

Proof. We may assume $\Phi(G_3) = 1$. It then follows from Theorem 1 that $G_4 \subseteqq P(G_2)$ and $G_5 \subseteqq P(G_3) = 1$. Also $P(G_2)$ is abelian, since

$$(P(G_2))_2 \subseteqq (P(G_2), G_2) \subseteqq P(G_4)G_{2(p+1)} = 1$$

by Lemma 1.

We next claim that $d(P(G_2)) \leqq d(G_2/G_3)$. For if $d(G_2/G_3) = t$, then there exist elements $g_1, \cdots, g_t$ in G_2 such that for each $g \in G_2$, $g = g_1^{m(1)} \cdots g_t^{m(t)}h$ for some integers $m(i)$ and $h \in G_3$. It follows from (*) that $g^p = (g_1^p)^{m(1)} \cdots (g_t^p)^{m(t)}h^p cd$, where h^p and c are elements of $P(G_3)$ and $d \in G_{2p}$. Hence $h^p = c = d = 1$ and the assertion follows.

Since $P(G_2)$ is abelian and $G_4 \subseteqq P(G_2)$ we thus have $d(G_4) \leqq d(G_2/G_3)$. Hence

$$\begin{aligned} d(G_3) &\leqq d(G_3/G_4) + d(G_4) \\ &\leqq d(G_3/G_4) + d(G_2/G_3) \\ &\leqq (1/2)d^2(d-1) + (1/2)d(d-1) , \end{aligned}$$

where the last inequality follows from Theorem 2.83 of [4].

THEOREM 7. *Suppose* $p > 3$ *and* $k \geqq 2$. *Let* $x_1, x_2, \cdots, x_d$ *be coset representatives of a minimal basis of the abelian group* $G_k/G_k^{(1)}$. *If* $G_3 \subseteqq P(G)$, *then there exist integers* $n(i)$ *such that*

$$(G_k)^{(1)} = \langle x_1^{p^{n(1)}}, \cdots, x_d^{p^{n(d)}} \rangle .$$

Proof. In any p-group, $(G_k)_2 \subseteqq G_{2k}$. Since $k \geqq 2$ it follows from Theorem 1 that $G_{2k} \subseteqq P(G_{2k-2}) \subseteqq P(G_k)$. Thus the theorem follows from Theorem 3 of [6].

COROLLARY 7.1. *Suppose* $G_3 \subseteqq P(G)$ *where* $p > 3$. *If* $k \geqq 2$ *and if* G_k *can be generated by* d *elements, then* $(G_k)^{(i)}$ *can be generated by* d *elements for* $i = 1, 2, 3, \cdots$.

A p-group G is called *p-abelian* if $(xy)^p = x^p y^p$ for all elements x, y of G. The properties of p-abelian groups used below may be found in [6] (p. 853).

THEOREM 8. *If* $p > 3$, $G_3 \subseteqq P(G)$, *and* $d = d(G)$, *then* $d(G^{(i)}) \leqq (1/2)d(d+1)$ *for* $i = 1, 2, 3, \cdots$.

Proof. We first consider the case $i = 1$. The result is clearly true in this case if $|G| = p$. Suppose the theorem is true when $i = 1$ for all groups H with $|H| < |G|$. We may assume $\Phi(G^{(1)}) = 1$. By Theorem 2.83 of [4], $d(G^{(1)}/G_3) \leqq (1/2)d(d-1)$. A p-group G is p-abelian modulo $P(G^{(1)})G_p$. Since $p > 3$, $G_p \subseteqq P(G_{p-2}) = 1$, so $P(G^{(1)})G_p = 1$ and G is p-abelian. Hence $d(P(G)) \leqq d$. In a p-abelian group $P(G) \subseteqq Z(G)$, so $P(G)$ is abelian. Since $G_3 \subseteqq P(G)$ we have $d(G_3) \leqq d$, so

$$d(G^{(1)}) < d(G^{(1)}/G_3) + d(G_3) \leqq (1/2)d(d+1)\,.$$

Thus the theorem is true for $i = 1$.

For $i > 1$, Corollary 7.1 yields.

$$d(G^{(i)}) = d((G^{(1)})^{(i-1)}) \leqq d(G^{(1)}) \leqq (1/2)d(d+1)\,.$$

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