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ON THE DECOMPOSITION OF INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS WITH CONTINUOUS POISSON SPECTRUM, II

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Let f be an infinitely divisible characteristic function whose spectral functions are absolutely continuous functions with almost everywhere continuous derivatives. Some necessary conditions that f belong to the class I_0 of the infinitely divisible characteristic functions without indecomposable factors have been obtained by Cramér, Shimizu and the author and a sufficient condition that f belong to I_0 has been given by Ostrovskiy. In the present work, we prove that the condition of Ostrovskiy is not only a sufficient, but also a necessary condition that f belong to I_0.

Let f be the function of the variable t defined by

$$\begin{aligned} \log f(t) &= \int_{-\infty}^{0} [e^{itu} - 1 - itu(1+u^2)^{-1}]\varphi(u)du \\ &\quad + \int_{0}^{\infty} [e^{itu} - 1 - itu(1+u^2)^{-1}]\psi(u)du \end{aligned} \tag{1}$$

where log means the branch of logarithm defined by continuity from $\log f(0) = 0$ and where φ and ψ are almost everywhere nonnegative and continuous functions which are defined respectively on $]-\infty, 0[$ and $]0, +\infty[$ and satisfy the condition

$$\int_{-\varepsilon}^{0} u^2\varphi(u)du + \int_{0}^{\varepsilon} u^2\psi(u)du < +\infty$$

for any $\varepsilon > 0$. If we let

$$M(x) = \int_{-\infty}^{x} \varphi(u)du \qquad x < 0 ,$$

$$N(x) = -\int_{x}^{+\infty} \psi(u)du \qquad x > 0 ,$$

then we see that the conditions of the Lévy representation theorem ([4], Th. 5.5.2) are satisfied, so that f is an infinitely divisible characteristic function. In [3], we have proved the following result.

If the two following conditions are satisfied:

(a) $\varphi(u) \geqq k$ a.e. for $-c(1+2^{-n}) < u < -c$,

(b) $\psi(u) \geqq k$ a.e. for $d < u < d(1+2^{-n})$,

where k, c and d are positive constants and n is a positive integer, then the function f defined by (1) has an indecomposable factor.

The following theorem completes this result.

THEOREM 1. *If*

$$\psi(u) \geqq k \text{ a.e. } for \ c < u < c(1+2^{-n}) \ and \ d < u < d(1+2^{-n})$$

where n is a positive integer and k, c and $d \geqq 2c$ are positive constants, then the function f defined by (1) *has an indecomposable factor.*

This theorem is an immediate consequence of the

LEMMA. *Let f be the characteristic function defined by*

$$\log f(t) = \int_0^\infty (e^{itu} - 1 - itu(1+u^2)^{-1})\alpha(u)du$$

where

$$\alpha(u) = \begin{cases} c & if \ 1 < u < \lambda \ or \ r < u < r\lambda \\ 0 & otherwise \end{cases}$$

c being a positive constant, $\lambda = 1 + 2^{-n}$ (n positive integer) and $r \geqq 2\lambda$. Then f has an indecomposable factor.

Proof. Let β be the function defined by

$$\beta(u) = \begin{cases} c & \text{if } 1 < u < \lambda \text{ or } r < u < r\lambda \\ -c\varepsilon & \text{if } \gamma < u < \delta \\ 0 & \text{otherwise} \end{cases}$$

$(2 < \gamma < \delta < 2\lambda)$ and α_m and β_m be the functions defined by

$$\alpha_1(x) = \alpha(x); \ \alpha_m(x) = \int_{-\infty}^\infty \alpha_{m-1}(x-t)\alpha_1(t)dt$$

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We prove easily by induction that

$$\beta_m(x) = \alpha_m(x) \geqq 0 \qquad \text{if } x \notin [A_m, B_m] \tag{2}$$

where A_m and B_m are defined by

$$A_m = m + 2^{-n}$$
$$B_m = mr\lambda - 2^{-n} .$$

We prove now that

(3) $$\lim_{\varepsilon\to 0}\sup_{A_m\leqq x\leqq B_m}|\alpha_m(x)-\beta_m(x)|=0\,.$$

Indeed, if $\varepsilon<1$, we have

$$|\alpha_m(x)|\leqq c^m(r\lambda-1)^{m-1}$$
$$|\beta_m(x)|\leqq c^m(r\lambda-1)^{m-1}$$

and from these formulae and from

$$\alpha_m(x)-\beta_m(x)=\int_{-\infty}^{\infty}[\alpha_{m-1}(x-t)\alpha_1(t)-\beta_{m-1}(x-t)\beta_1(t)]dt$$
$$=\int_{-\infty}^{+\infty}\alpha_{m-1}(x-t)[\alpha_1(t)-\beta_1(t)]dt-\int_{-\infty}^{+\infty}[\beta_{m-1}(x-t)-\alpha_{m-1}(x-t)]\beta_1(t)dt$$

it follows by induction that

$$|\alpha_m(x)-\beta_m(x)|\leqq\varepsilon(2c)^m(r\lambda-1)^{m-1}$$

and this implies (3).

Let now $S(\alpha_m)$ be the spectrum of α_m. From the definition of α_m, it follows easily that

$$S(\alpha_m)=\bigcup_{j=0}^{m}[j+(m-j)r,(j+(m-j)r)\lambda]\,.$$

This implies that $S(\alpha_m)$ is all the interval $[m, mr\lambda]$ if

$$m>K=[(r-1)(2^n+1)]$$

(here $[x]$ means the integer part of x) and therefore

(4) $$\inf_{A_m\leqq x\leqq B_m}\alpha_m(x)>0\qquad m=K+2,K+3,\cdots\,.$$

From (3) and (4), it follows that

(5) $$\beta_m(x)\geqq 0\qquad m=K+2,K+3,\cdots,2K+3$$

if ε is small enough. But, from the definition of β_m, we have for $k<m$

$$\beta_m(x)=\int_{-\infty}^{\infty}\beta_{m-k}(x-t)\beta_k(t)dt$$

so that, from (5)

(6) $$\beta_m(x)\geqq 0\qquad m\geqq K+2$$

if ε is small enough.

We consider now β_m for $m\leqq K+1$. β_m can be negative only on intervals of the kind

$$I = [j + kr + l\gamma, (j + kr)\lambda + l\delta]$$

where j and k are nonnegative integers and l a positive integer satisfying

$$j + k + l = m$$

and on I we have

$$|\beta_m(x)| \leqq \varepsilon c^m (r\lambda - 1)^{m-1} .$$

But we have

$$j + 2l + kr < j + kr + l\gamma < (j + kr)\lambda + l\delta < (j + 2l + kr)\lambda$$

so that α_{m+l} is positive on I. Therefore, using (3), we have

$$\sum_{\substack{1 \leqq j \leqq k+1 \\ j \neq m+l}} \frac{\beta_j(x)}{j!} + \frac{\beta_{m+l}(x)}{(m+l)!} > 0$$

for $x \in I$ if ε is small enough. This implies that

$$\sum_{j=1}^{2K+2} \frac{\beta_j(x)}{j!} \geqq 0$$

for any x and therefore from (6)

$$\sum_{j=1}^{\infty} \frac{\beta_j(x)}{j!} \geqq 0 \tag{7}$$

for any x if $\varepsilon > 0$ is small enough.

Let now g be the function defined by

$$\log g(t) = \int_{-\infty}^{\infty} (e^{itu} - 1 - itu(1 + u^2)^{-1})\beta(u)du .$$

Then

$$g(t) = \int_{-\infty}^{\infty} e^{itx} dG(x)$$

where G is the function

$$G(x) = e^{-\lambda}\left\{\chi(x + \eta) + \int_{-\infty}^{x} \left[\sum_{n=1}^{\infty} \frac{\beta_n(y + \eta)}{n!}\right] dy\right\} .$$

Here χ is the degenerate distribution and λ and η are defined by

$$\lambda = \int_{-\infty}^{\infty} \beta(u)du$$

$$\eta = \int_{-\infty}^{\infty} u(1 + u^2)^{-1}\beta(u)du .$$

From (7), it follows that g is a characteristic function if ε is small enough. Since g is not infinitely divisible, from the Khintchine's theorem ([4], Th. 6.2.2), g has an indecomposable factor and since g divides f, the lemma is proved.

As consequences of the Theorem 1, we obtain the following results which are respectively the results of Cramér [1] and Shimizu [6] quoted in the introduction.

COROLLARY 1. *If in an interval* $[0, r]$ $(r > 0)$, $\psi(u) \geqq c > 0$ *almost everywhere, then the function* f *defined by* (1) *has an indecomposable factor.*

COROLLARY 2. *If in an interval* $[r, s]$ $(s > 2r > 0)$, $\psi(u) \geqq c > 0$ *almost everywhere, then the function* f *defined by* (1) *has an indecomposable factor.*

The characterization announced in the introduction is the following.

THEOREM 2. *A necessary and sufficient condition that the function* f *defined by* (1) *has no indecomposable factor is the existence of an* $r > 0$ *such that one of the two following conditions is satisfied*:

(a) $\varphi(u) \equiv 0$ a.e.; $\quad \psi(u) = 0$ a.e. *if* $u \notin [r, 2r]$;

(b) $\psi(u) \equiv 0$ a.e.; $\quad \varphi(u) = 0$ a.e. *if* $u \notin [-2r, -r]$.

Proof. The sufficiency is a consequence of the Theorem 1 of Ostrovskiy [4] (see also [2], Th. 8.2), while the necessity follows immediately from the preceding theorem and from the Theorem 1 of [3] stated above.

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