A CHARACTERIZATION OF REGULAR MAXIMAL IDEALS

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In 1967, Gleason characterized the maximal ideals in a
commutative Banach algebra A with identity as those subspaces
M of codimension one in A which contain no invertible ele-
ments. (Kahane and Zelazko gave the same characterization
in 1968.) An equivalent statement of this property of M is:
(1)   Each element of M belongs to some regular maximal ideal.
The question we examine is when does this distinguish
the regular maximal ideals from the other subspaces of
codimension one in a commutative Banach algebra without
identity.

We show that if A is generated by a single element then
a closed subspace M of codimension one in A and satisfying
(1) is a regular maximal ideal and we show by an example
that this result may fail for an algebra which is generated
two elements. We have results related to the above, which
can be applied to $L^1(G)$, where G is a locally compact abelian
metrizable group; a sample corollary is that a (not a priori
closed) subspace M of codimension one for which (1) holds is
a regular maximal ideal in $L^1(G)$.

That property (1) fails to characterize regular maximal ideals in
an arbitrary commutative Banach algebra can be easily seen by con-
sidering a locally compact Hausdorff space S which is not sigma-
compact, i.e., which is not a countable union of compact sets, and
the Banach algebra $C_0(S)$ of continuous functions vanishing at infinity
on S taken with the supremum norm. For any f in $C_0(S)$, $\{s: f(s) \neq 0\}$
is sigma-compact and so f belongs to infinitely many regular maximal
ideals in $C_0(S)$. Consequently any subspace M of codimension one in
$C_0(S)$ has property (1) and yet need not be a regular maximal ideal.
This example points out that we should consider only algebras in
which some element belongs to no regular maximal ideal.

Lemma 1. *Let A be a commutative Banach algebra and x_0 an
element belonging to no regular maximal ideal. Suppose that M is
a subspace of codimension one in A with the property that each
element in M belongs to some regular maximal ideal and let x^* be
a nonzero linear functional on A, not necessarily continuous, which
vanishes (precisely) on M. Then for all a and b in A,*

(2) $$x^*(x_0)x^*(abx_0) = x^*(ax_0)x^*(bx_0) .$$

Proof. We adjoin an identity e to A in the usual way [6, p. 59]

obtaining the Banach algebra A_e of pairs (a, λ) where a is in A and λ is a complex number. Recall that $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu)$, $e = (0, 1)$ and $\| (a, \lambda) \| = \| a \| + | \lambda |$.

Define the subspace M_0 of A_e by: $M_0 = \{(a, \lambda) : (a, \lambda)(x_0, 0)$ is in $M \times \{0\}\}$. Let (a, λ) in A_e be given and set $\mu = \lambda + (x^*(ax_0)/x^*(x_0))$, noting that $x^*(x_0) \neq 0$ since x_0 is not in M, and consider $(a, \lambda) - \mu e$. Because $x^*(ax_0 + (\lambda - \mu)x_0) = 0$, $ax_0 + (\lambda - \mu)x_0$ is in M and so $A_e = M_0 + sp(e)$.

We now show that M_0 contains no invertible elements. Suppose that (a, λ) is an invertible element in M_0 with inverse (b, μ). Then

$$(3) \qquad ab + \lambda b + \mu a = 0 ,$$

$$(4) \qquad \lambda \mu = 1, \text{ and}$$

$$(5) \qquad ax_0 + \lambda x_0 \text{ belongs to } M .$$

From (5) there is a point s in the regular maximal ideal space of A with $\hat{a}(s)\hat{x}_0(s) + \lambda \hat{x}_0(s) = 0$ and thus $\hat{a}(s) = -\lambda$. Evaluating expression (3) at s we obtain $\hat{a}(s)\hat{b}(s) + \lambda \hat{b}(s) + \mu \hat{a}(s) = -\lambda \mu = 0$, which contradicts (4).

Applying Gleason's characterization [3], M_0 must be a maximal ideal in A_e. Let a and b be in A and choose $\lambda = -x^*(ax_0)/x^*(x_0)$. Then (a, λ) is in M_0 and consequently $(a, \lambda)(b, 0)$ is in M_0, i.e., $x^*(abx_0 + \lambda bx_0) = 0$ which yields equation (2).

THEOREM 2. *Let A be a commutative Banach algebra generated by the element x_0. If a closed subspace M of codimension one in A has the property that each element in M belongs to some regular maximal ideal then M is a regular maximal ideal.*

Proof. Let z and w be in A. Since Ax_0 is a dense ideal there is a sequence $\{a_n\}$ with $w = \lim a_n x_0$. Let x^* be a nonzero continuous linear functional on A with $x^*(M) = 0$. Using equation (2), $x^*(zw) = \lim x^*(a_n zx_0) = \lim x^*(a_n x_0)x^*(zx_0)/x^*(x_0) = x^*(w)x^*(zx_0)/x^*(x_0)$. Thus if w is in M, zw must be in M.

EXAMPLE 3. We display a commutative Banach algebra A generated by two elements (and therefore with a sigma-compact regular maximal ideal space) with the property that every element in it belongs to infinitely many regular maximal ideals. For this algebra, any subspace of codimension one has property (1) and so (1) does not characterize regular maximal ideals in A. The algebra A further shows that a subspace of codimension one in A having property (1) need not be closed; we ask whether this is possible for an algebra

with a single generator. For the example let S be the bicylinder, $\{(\lambda, \mu): \lambda \text{ and } \mu \text{ complex numbers with } |\lambda| \leqq 1 \text{ and } |\mu| \leqq 1\}$. Let P be the algebra of all polynomials in the two variables λ and μ which vanish at $(0, 0)$ and let A be the closure of P in $C(S)$. A is generated by λ and μ. Any nonzero multiplicative linear function x^* on A has norm not exceeding one and so the scalars $x^*(\lambda) = \lambda_0$ and $x^*(\mu) = \mu_0$ each have absolute value at most one. Thus the pair (λ_0, μ_0) lies in $S - \{(0, 0)\}$, and for each f in A, $x^*(f) = f(\lambda_0, \mu_0)$. That is, the regular maximal ideal space of A can be identified with $S - \{(0, 0)\}$. Any function f in A is an analytic function in two complex variables on the interior of S [2, Theorem 9.12.1, p. 229} which vanishes at $(0, 0)$ and consequently [2, Problem 4, p. 245] vanishes at infinitely many points in S. The algebra A is then the desired example.

A commutative Banach algebra A has an *approximate identity* if there is a generalized sequence $\{u_\alpha\}$ of elements of A with $a = \lim u_\alpha a$ for all a in A, and has a *bounded approximate identity* if, in addition, the elements u_α are uniformly bounded in norm. This usage is not standard; one usually discusses only a bounded approximate identity which is then called an approximate identity. An interesting Banach algebra having an unbounded approximate identity is the algebra $L^1(G) \cap G^2(G), G$ a locally compact abelian topological group [9, Definition 4.6, p. 411]. We point out that one cannot generally modify an unbounded approximate identity to get a bounded one. In fact if a Banach algebra A has an unbounded approximate identity $\{u_n\}$ which is a sequence then it cannot have a bounded approximate identity $\{v_\alpha\}$. To see this, one observes that the operators T_n defined $T_n x = u_n x$ are pointwise convergent. Hence, by the Uniform Boundedness Theorem, $\sup \| T_n \|$ is finite, so that $\| u_n \| = \lim_\alpha \| T_n v_\alpha \| \leqq \sup \| T_n \| \sup \| v_\alpha \|$.

THEOREM 4. *Let A be a commutative Banach algebra with an approximate identity each element of which belongs to no regular maximal ideal, for example A a Banach algebra with an approximate identity, a continuous involution $x \rightarrow \tilde{x}$ with $\tilde{x}^{\wedge} = \tilde{x}^-$, and a sigma-compact regular maximal ideal space. Then a closed subspace M of codimension one in A with the property that each element in M belongs to some regular maximal ideal must be a regular maximal ideal.*

Proof. First we show that the hypotheses are satisfied by A having an approximate identity $\{u_\alpha\}$, an involution $x \rightarrow \tilde{x}$, as above, and a sigma-compact regular maximal ideal space. Because the involution is continuous $\{\tilde{u}_\alpha\}$ is also an approximate identity and the elements $w_\alpha = (u_\alpha + \tilde{u}_\alpha)/2$ form an approximate identity with $\hat{w}_\alpha$ real-valued. The algebra A is selfadjoint [6, p. 89] with a sigma-compact regular maximal ideal space and so there is an element v in A with

$\hat{v}(s)$ strictly greater than zero for each s in $\Delta(A)$. The generalized sequence with entries $v_{\alpha,n} = w_\alpha + iv/2^n$ and the usual product ordering forms an approximate identity for A and each $v_{\alpha,n}$ belongs to no regular maximal ideal.

Now suppose that A has an approximate identity $\{e_\alpha\}$, no e_α belonging to a regular maximal ideal, and suppose that x^* is a nonzero continuous linear functional on A with $x^*(M) = 0$. Using Lemma 1, for each a and b in A we have

$$(6) \qquad x^*(e_\alpha)x^*(abe_\alpha) = x^*(ae_\alpha)x^*(be_\alpha) .$$

By passing to a subnet we may suppose that $\{x^*(e_\alpha)\}$ converges, either to a complex number β or to infinity. If $\{x^*(e_\alpha)\}$ converges to infinity, then from (6) we see that $x^*(ab) = 0$ for all a and b in A and consequently $x^*(a) = \lim x^*(ae_\alpha) = 0$ for each a, a contradiction. If $\{x^*(e_\alpha)\}$ converges to $\beta = 0$, then from (6) x^* is again zero. Thus $\{x^*(e_\alpha)\}$ converges to a nonzero complex number β. By taking limits in (6) we conclude x^*/β is a multiplicative linear functional and therefore that M is a regular maximal ideal.

THEOREM 5. *Let A be a commutative Banach algebra with a bounded approximate identity $\{u_\alpha\}$ having $\operatorname{Re} \hat{u}_\alpha(s) \gneqq 0$ for all s in the regular maximal ideal space of A and all α, for example A a semi-simple commutative algebra with a bounded approximate identity and a sigma-compact regular maximal ideal space. Then a subspace M of codimension one in A, which is not a priori closed, having the property that each element in M belongs to some regular maximal ideal must be a regular maximal ideal.*

Proof. To see that a semi-simple Banach algebra A with a bounded approximate identity and a sigma-compact maximal ideal space has an approximate identity of the desired type we use some recent results of D.C. Taylor. Under these circumstances A has a bounded approximate identity $\{v_\alpha\}$ with $\operatorname{Re} \hat{v}_\alpha(s) \geqq 0$ for all s and α [8, Corollary 2.3] and contains an element v with $\operatorname{Re} \hat{v}(s) \gneqq 0$ for all s in $\Delta(A)$ [8, Th. 1.1]. The generalized sequence $\{v_\alpha + v/2^n\}$ under the product ordering is an approximate identity of the desired type.

Let A have a bounded approximate identity $\{u_\alpha\}$ as hypothesized and suppose that x^* is a nonzero linear functional, not a priori continuous, with $x^*(M) = 0$. Let b and c in A be given. By Paul Cohen's factorization theorem [1, Th. 1] and by examining the conclusion of the proof on page 202 it follows that $c = ax_0$ where $x_0 = \sum \gamma(1 - \gamma)^{n-1}u_{\alpha_n}$, $0 < \gamma < \frac{1}{4}$, and each u_{α_n} a member of $\{u_\alpha\}$. Since $\operatorname{Re} \hat{x}_0(s) \gneqq 0$ for all s in $\Delta(A)$ we can apply Lemma 1 to conclude that

$$x^*(x_0)x^*(bc) = x^*(x_0)x^*(abx_0) = x^*(ax_0)x^*(bx_0) = x^*(c)x^*(bx_0) \ .$$

Hence if c is in M, bc is in M and M is a maximal ideal. To complete the proof it will suffice to show that the maximal ideals of A are all regular. Since $A^2 = A$ [1], this follows from Theorem 4.0b and Corollary 4.0.3 in [7, pp. 87–88].

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