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ELLIPTIC DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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The purpose of this paper is to furnish a proof of the following theorem:

THEOREM. D_1 **and** D_2 **are two disjoint open sets in the** xy**-plane having the open arc** σ **as a common boundary.** L_i **in** D_i, $i = 1, 2$, **are defined as**

$$L_i(\phi) \equiv a_i\phi_{xx} + 2b_i\phi_{xy} + c_i\phi_{yy} + d_i\phi_x + e_i\phi_y + g_i\phi,\ a_ic_i - b_i^2 > 0 .$$

Functions u_i **satisfy** $L_i(u_i) = f_i$ **in** D_i, **with** $u_i \in C^2$ **in** D_i **and** $\in C^1$ **in** $D_iU\sigma$; **on** $\sigma, u_1 = u_2$ **and** $\partial u_1/\partial N_1 = k(s)\partial u_2/\partial N_2$, **where** s **is arc length on** σ, $k(s) > 0$, **and** $\partial u_i/\partial N_i$ **denotes the conormal derivative of** u_i. **If, on** $D_iU\sigma, a_i, b_i, c_i \in C_\alpha^{n+2}$; $d_i, e_i, g_i, f_i \in C_\alpha^n$; $k \in C_\alpha^{n+2}$ **and** $\sigma \in C_\alpha^{n+3}$; **then** $u_i \in C^{n+2}$ **on** $D_iU\sigma$ **for** $n \geqq 0$. **If all indicated quantities are analytic functions of their arguments and** σ **is an analytic arc, then** u_i **is analytic on** $D_iU\sigma$.

Here $u \in C_\alpha^n$ on G means the nth order derivatives of u satisfy a uniform Hölder condition with exponent α on every compact subset of G. The conormal derivative

$$\partial u_i/\partial N_i = (a_ir + b_is)u_x + (b_ir + c_is)u_y,\ r^2 + s^2 = 1 ,$$

uses the same unit normal (r, s) to σ for $i = 1$ or 2; this normal may point into D_1 or it may point into D_2.

Such elliptic equations occur in physical problems involving continuous media of different properties. Smoothness of solutions plays an important role in the numerical analysis of such problems [7], [6].

Oleinik [5] has investigated the smoothness of solutions to such problems in several dimensions starting with weaker differentiability hypotheses on the u_i. This work differs from hers in that here analyticity is proved, and no restriction on g_i is required for uniqueness of solutions is not required in the proof. The proof here is also along different lines since the restriction to two dimensions allows the use of conformal mapping and the Beltrami differential equation to bring L_i to normal form, allowing previous results of the author [8] to be applied.

2. Since the proof of the theorem is based on coordinate transformations it is essential to examine how the various coefficients and the problem as a whole are altered under point transformations. With the symbolism

$$
(1)\qquad \nabla_{xy} = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}, \qquad J = \begin{pmatrix} \phi_x & \psi_x \\ \phi_y & \psi_y \end{pmatrix}, \qquad H = \begin{pmatrix} a & b \\ b & c \end{pmatrix},
$$

$$
\nabla_{xy} = J\nabla_{\xi\eta}, \qquad J \text{ nonsingular}, \qquad A^T \equiv \text{transpose of } A,
$$

the second-order terms (principal part) in

$$
(\nabla_{xy}^T H \nabla_{xy})u
$$

give $au_{xx} + 2bu_{xy} + cu_{yy}$. Under the transformation $\xi = \phi(x, y)$, $\eta = \psi(x, y)$,

$$
(2)\qquad \nabla_{xy}^T H \nabla_{xy} = (J\nabla_{\xi\eta})^T HJ\nabla_{\xi\eta}\,.
$$

Thus the principal part of the right side in $\xi - \eta$ coordinates is associated with the matrix J^THJ. An arc σ in the xy-plane $h(x, y) = 0$ become $h^*(\varepsilon, \eta) = 0$ and the normals $\nabla_{xy}h$ and $\nabla_{\xi\eta}h^*$ satisfy

$$
\nabla_{xy}h = J\nabla_{\xi\eta}h^*\,.
$$

Thus the conormal direction in the xy-plane

$$
(3)\qquad \frac{1}{\|\nabla h\|} H\nabla h = N
$$

becomes in the $\xi\eta$ plane

$$
N^* = \frac{1}{\|\nabla h^*\|} J^THJ\nabla h^* = \frac{1}{\|J^{-1}\nabla h\|} J^TH\nabla h\,.
$$

And the conormal derivative $N^T\nabla_{xy}u$ becomes $(N^*)^T\nabla_{\xi\eta}u$ with

$$
(4)\qquad \begin{aligned} (N^*)^T\nabla_{\xi\eta}u &= \frac{1}{\|J^{-1}\nabla h\|}(\nabla h)^TH^TJJ^{-1}\nabla_{xy}u \\ &= \frac{\|\nabla h\|}{\|J^{-1}\nabla h\|} N^T\nabla_{xy}u \end{aligned}
$$

where $\|\;\|$ is the usual Euclidean length of a vector.

Under a transformation of class C_α^n with nonvanishing Jacobian an arc of class C_α^n is transformed into an arc of class C_α^n and corresponding arc length parameters are connected by an invertible transformation of class C_α^n, if $n \geqq 1$.

Thus in the theorem, if $D_1UD_2U\sigma$ is subjected to a (nonsingular) transformation of class C_α^{n+3}, the problem transforms into a problem of the same type. A similar statement holds in the analytic case if the transformation is analytic.

The plan of proof is to map σ into the x-axis and the principal parts of L_i into Laplacian operators, then apply the results of [8]. In [8], the theorem has been proved for Laplacian principal parts for k constant, but the proof carries through for k variable, though the second derivatives of k will appear in the coefficients of first order derivative terms of the modified equations.

The theorem is proved for a neighborhood of a point on σ; then a Heine-Borel argument extends the result to a neighborhood of σ.

3. If σ is analytic and p_0 is a point on σ, there is a conformal map (nonsingular) taking p_0 into the origin and a neighborhood of p_0 onto a neighborhood of the origin in such a way that the points on σ are mapped onto the x-axis. Since the mapping is one to one and analytic both ways, the problem becomes one of proving the analytic case of the theorem when σ is the x-axis and D_1 is the region $y > 0$, D_2 the region $y < 0$.

As is known, $L(u) = au_{xx} + 2bu_{xy} + cu_{yy}$ may be brought to a form with Laplacian principal part by a transformation $\xi = \phi(x, y)$ $\eta = \psi(x, y)$ if ϕ, ψ satisfy the Beltrami system

$$\begin{aligned} \phi_x &= \frac{b\psi_x + c\psi_y}{\delta} \\ \phi_y &= -\frac{a\psi_x + b\psi_y}{\delta} \end{aligned} \qquad \delta^2 = ac - b^2 > 0 \tag{5}$$

and are at least twice continuously differentiable. Indeed

$$L(u) = (a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2)(\nabla^2_{\xi\eta}u) + \text{lower terms}\ , \tag{6}$$

where the lower terms involve the second derivatives of ψ and first derivatives of u.

With $\delta_i^2 = a_ic_i - b_i^2 > 0$ and the notation of the theorem, we solve the initial value problems

$$\begin{aligned} &\frac{\partial}{\partial x}\Big(\frac{a_i\psi_{ix} + b_i\psi_{iy}}{\delta_i}\Big) + \frac{\partial}{\partial y}\Big(\frac{b_i\psi_{ix} + c_i\psi_{iy}}{\delta_i}\Big) = 0 \\ &\text{with } \psi_i(x, 0) = 0,\ \psi_{iy}(x, 0) = \frac{\delta i}{c_i}\ , i = 1, 2\ . \end{aligned} \tag{7}$$

Since a_i, b_i, c_i, δ_i are all analytic, (7) is an elliptic second order linear equation with analytic coefficients with prescribed analytic initial data. By the Cauchy-Kowaleski theorem there exists in a neighborhood of the x-axis a unique analytic ψ_i satisfying the initial value problem. Both ψ_1 and ψ_2 are two-sided solutions since the a_i, etc., are analytic in a two-sided neighborhood of σ. From (7) it follows that the right-hand side of (5) are components of the gradient of a scalar ϕ, which may be constructed from the gradient by a line integral

$$\phi_i(x, y) = \int_{(0,0)}^{(x,y)} \frac{b_i\psi_{ix} + c_i\psi_{iy}}{\delta_i} dx - \frac{a_i\psi_{ix} + b_i\psi_{iy}}{\delta_i} dy \,. \tag{8}$$

Since by (7) the integral is independent of path, the path for computing $\phi_i(x, 0)$ is the x-axis. Since $\psi_{ix}(x, 0) = 0$ from the initial values, $\phi_i(x, 0) = x$, $i = 1, 2$.

The transformation $\xi = \phi_i(x, y)$, $\eta = \psi_i(x, y)$, x, y in $D_i U \sigma$ maps a neighborhood of the origin in the xy-plane onto a neighborhood of the origin in the $\xi\eta$-plane with the x-axis mapped onto the ξ-axis. The portion $y > 0$ is mapped onto $\eta > 0$; and $y < 0$ onto $\eta < 0$. The Jacobian

$$\begin{vmatrix} \phi_{ix} & \phi_{iy} \\ \psi_{ix} & \psi_{iy} \end{vmatrix} = \begin{vmatrix} \dfrac{c_i}{\delta_i}\psi_{iy} & \phi_{iy} \\ 0 & \psi_{iy} \end{vmatrix} = \frac{\delta_i}{c_i} > 0$$

on $y = 0$ and, therefore, the mapping in each region, analytically continuable into the other, is invertible, the inverse being analytic. The problem in the theorem is now reduced to one in which the equations have Laplacian principal parts (after division by $(a_i\psi_{ix}^2 + 2b_i\psi_{ix}\psi_{iy} + c_i\psi_{iy}^2)$ as indicated in (6) with $i = 1$ for $\eta > 0$, $i = 2$ for $\eta < 0$) and σ is the ξ-axis. By [8], the solutions have the analyticity property on the ξ-axis, i.e., the solution in the region $\eta > 0$ is analytically continuable into the lower region $\eta < 0$ and conversely. Since every mapping from the original problem to this canonical problem has nonvanishing Jacobian on σ, the inverses are also analytic in a neighborhood of p_0 on σ and, therefore, the solutions of the equations have the analyticity property on σ.

Since with each point on σ there is a neighborhood of the point in which the solutions are analytic, it is possible to cover any closed subarc of σ with a finite number of such neighborhoods which overlap. The solutions are analytic in the union of these neighborhoods.

4. In the nonanalytic case of the theorem, there is a mapping of class C_α^{n+3} with nonvanishing Jacobian taking p_0 on σ into the origin and a neighborhood of p_0 into a neighborhood of the origin in such a way that σ maps onto the x-axis with D_1 mapped into $y > 0$ and D_2 into $y > 0$. This is by definition of $\sigma \in C_\alpha^{n+3}$. Thus, as before, we may assume in the theorem that σ is the x-axis. Again we use the Beltrami system to bring $L_i(u)$ into Laplacian principal part by solving (7). (7) is a linear second order elliptic equation with coefficients in C_α^{n+1}. However, the Cauchy-Kowaleski theorem cannot be used to establish the existence of the desired mapping because the data is not analytic. Instead, the Schauder theory gives the existence.

We let K_1 be a curve in $D_1U\sigma$ of class C_α^{n+3} coinciding with a segment of the x-axis containing the origin in the middle of the segment. From the Schauder theory there exists $\psi_1(x, y)$ of class C_α^{n+3} in the closed region bounded by K_1 satisfying (7) and assuming on K_1 the boundary values $\psi_1 = y$. By Hopf's theorem [2], $\psi_{1y}(x, 0) > 0$. The ϕ in (5) is again reconstructed from its gradient by a line integral as in (8), the curve being confined to the closure of the region bounded by K_1. The Jacobian of the transformation $\xi = \phi_1(x, y)$, $\eta = \psi_1(x, y)$ at $x = 0, y = 0$ is

$$\begin{vmatrix} \phi_{1x} & \phi_{1y} \\ \psi_{1x} & \psi_{1y} \end{vmatrix} = \phi_{1x}\psi_{1y} = \frac{c_1}{\delta_1}\psi_{1y}^2 > 0 .$$

The function pair ϕ_1, ψ_1 can be extended into D_2 as class C_α^{n+3} functions [1]. (This can easily be done for $x \in K_1 \cap \sigma$. If

$$h(x, y) = \sum_{j=0}^{n+3} \frac{\partial^j \psi_1(x, 0)}{\partial y^j} \frac{y^j}{j!}$$

then $v(x, y) = \psi_1(x, y) - h(x, y)$ vanishes with its y derivatives of order $\leqq n + 3$ for $y = 0$. Thus v defined in D_1 is easily extended into D_2 by $v(x, -y) = v(x, y)$. If $V(x, y)$ is the extended v, $V \in C_\alpha^{n+3}$ in $D_1U\sigma UD_2$. $\psi(x, y) = V(x, y) + h(x, y)$ is therefore the desired extension of ψ_1.) Therefore, there is a class C_α^{n+3} transformation mapping a neighborhood of the origin in such a way that the x-axis goes into the ξ-axis, L_1 transforms into an operator which has principal part equal to a (nonvanishing) scalar times the Laplacian, as in (6), and the Jacobian does not vanish in the neighborhood. L_2 is also transformed into a linear second order elliptic operator. Both new equations are now divided by the coefficient of the Laplacian. Thus the theorem is now reduced to the case in which L_1 is the Laplacian and σ is the x-axis.

We now use the Beltrami system to bring the current L_2 to Laplacian principal part, proceeding initially as above to obtain $\phi_2(x, y)$ and $\psi_2(x, y)$ in $D_2U\sigma$ such that $\psi_2(x, 0) = 0$. Then the following transformation is applied to a neighborhood of the origin:

$$(9) \qquad \left.\begin{aligned} \xi &= \phi_2(x, y) \\ \eta &= \psi_2(x, y) \end{aligned}\right\} y \leqq 0 \, ; \qquad \left.\begin{aligned} \xi &= \phi_2(x, 0) \\ \eta &= y \end{aligned}\right\} y \geqq 0 \, .$$

The mapping of D_1 has a C_α^{n+3} extension into D_2 and the mapping of D_2 has a C_α^{n+3} extension into D_1. Both mappings agree on $y = 0$ and have nonvanishing Jacobian at (0, 0).

$$L_1(u_1) = [c_2(x, 0)\psi_{2y}(c, 0)/\delta_2(x, 0)]^2 u_{1\xi\xi} + u_{1\eta\eta}$$
$$+ \text{ derivatives of } u_1 \text{ of lower order}$$

$$L_2(u_2) = (a_2\psi_{2x}^2 + 2b_2\psi_{2x}\psi_{2y} + c_2\psi_{2y}^2)(\nabla_{\xi\eta}^2 u_2)$$
$$+ \text{ derivatives of } u_2 \text{ of lower order .}$$

Thus

$$\begin{aligned} L_1(u_1) &\equiv h^2(\xi)u_{1\xi\xi} + u_{1\eta\eta} + \cdots \\ L_2(u_2) &\equiv g(\xi, \eta)\nabla_{\xi\eta}^2 u_2 + \cdots \end{aligned} \tag{10}$$

where h and g are positive and $\in C_\alpha^{n+2}$. On account of (9) and (4), the $k(\xi)$ for (10) is still of class C_α^{n+2}. (It is at this point that C_α^{n+3} transformations are required, rather than C_α^{n+2} because of the degrading of the differentiability of k at this step by introducing factors depending on the first derivatives of the transformation.) Division of the upper equation in (10) by $h^2(\xi)$ and the lower one by $g(\xi, \eta)$ yields

$$\begin{aligned} L_1(u_1) &= u_{1\xi\xi} + q^2(\xi)u_{1\eta\eta} + \cdots \\ L_2(u_2) &= \nabla_{\xi\eta}^2 u_2 + \cdots \end{aligned} \tag{11}$$

the dots indicating terms involving u derivatives of order less than two. The $k(\xi)$ for (11) is just the k for (10) multiplied by $g(\xi, 0)/h^2(\xi)$. Thus the theorem has been reduced to the case (11) with σ the ξ-axis.

Returning to $x - y$ variables to avoid excessive notation we make the variable change $x = \xi$, $y = \eta/q(\xi)$ for $\eta \geqq 0$, $x = \xi$, $y = \eta$ for $\eta \leqq 0$. This leaves L_2 unchanged, and brings L_1 into

$$L_1(u_1) = u_{1xx} + 2r(x)yu_{1xy} + (r^2y^2 + 1)u_{1yy} + \cdots \tag{12}$$

where $r(x) = -q'/q$. Thus the coefficients in the principal part of L_1 meet with those of L_2 to make them Lipschitz continuous throughout $C(0, a)$, a circle in $D_1 U \sigma U D_2$ centered at $(0, 0)$ with radius a.

As in [8] we write $u_1 = u + kw$, $u_2 = u + w$ where u is even, w odd, and both $\in C^1$ in $C(0, a)$. This leads to an elliptic system as in [8], but somewhat more complicated. However, the coefficients of second order derivative terms are Lipschitz continuous, and as in (12), the x-derivative of those coefficients are Lipschitz in $C(0, a)$. By Theorem 4.3 in [3], $u, w \in C_\alpha^1$ in $C(0, a)$. Now the proof of Theorem 4.5 in [3] can be applied, modified to use only derivatives and difference quotients in the x-direction, noting that the v in that proof is of class C_α^1 with one even component, one odd, but that in any case $v_x \in C_\alpha^1$. That proof puts u_x, w_x in C_α^1 in $C(0, a)$. u_{yy} and w_{yy}, for $y \geqq 0$, are expressed in terms of quantities which are of class C_α from the differential equations. Thus, for $y \geqq 0$, u and w are of class C_α^2.

The further differentiability properties on u and w are obtained by differentiating the equations with respect to x (using always the known interior differentiability properties) and observing that u_x, w_x satisfy an elliptic system like u and w with the same boundary conditions on $y = 0$. Thus $u_x, w_x \in C_\alpha^2$ for $y \geqq 0$. The equations for $y \geqq 0$ are then differentiated with respect to y and solved for u_{yyy} and w_{yyy} in terms of quantities which are class C_α. The process may be repeated as often as is possible to differentiate the coefficients. The differentiability properties of u_i then follow.

This result gives three orders of differentiability more than the results in [5].

5. The theorem is valid if the solution pair u_1, u_2 are initially known only to be in H_2^1. (Classes H_p^m are defined in [4], pp. 62–63.) By this, here, is meant that for any closed subset S^* of $S = D_1 U \sigma U D_2$ there is a sequence of function pairs (u_1^j, u_2^j), $u_i^j \in C^1$ in $D_i U \sigma$, satisfying the conormal derivative condition of the theorem, such that u_i^j converges in the mean of order two to u_i on $S^* \cap D_i$ while the first order derivatives form a Cauchy sequence in mean of order two on $S^* \cap D_i$.

The proof will be merely sketched for the analytic case. The nonanalytic case is similar. Clearly the H_2^1 hypotheses hold for the transformed problem, so it is sufficient to prove the result for the canonical form above, as in [8]. As there, $u_2 = u + v$, $u_1 = u + kv$ where u is even, v is odd; and u, v satisfy in S, which may be taken as the open circle $C(0, a)$ centered at $(0, 0)$ with radius a,

$$\int_{R^*} \frac{\partial u}{\partial n} ds = \int_R F_1 dxdy \ , \qquad \int_{R^*} \frac{\partial v}{\partial n} ds = \int_R F_2 dxdy$$

for almost all closed cells R in $C(0, a)$ [3]. Here F_1 and F_2 are linear combinations of u, v, f_1, f_2, and the first order derivatives of u and v as in [8] with bounded coefficients. By Theorem 4.3 in [3], u and v are of class C_α^1 in $C(0, a)$. Thus u_i is of class C_α^1 on $D_i U \sigma$. The further results then follow.

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