PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 2, 1969

CHARACTERIZATION OF CERTAIN INVARIANT SUBSPACES OF H^p AND L^p SPACES DERIVED FROM LOGMODULAR ALGEBRAS

SAMUEL MERRILL, III, AND NAND LAL

Let $A = A(X)$ be a logmodular algebra and m a representing measure on X associated with a nontrivial Gleason part. For $1 \leqq p \leqq \infty$, let $H^p(dm)$ denote the closure of A in $L^p(dm)$ (w^* closure for $p = \infty$). A closed subspace M of $H^p(dm)$ or $L^p(dm)$ is called *invariant* if $f \in M$ and $g \in A$ imply that $fg \in M$. The main result of this paper is a characterization of the invariant subspaces which satisfy a weaker hypothesis than that required in the usual form of the generalized Beurling theorem, as given by Hoffman or Srinivasan.

For $1 \leqq p \leqq \infty$, let I^p be the subspace of functions in $H^p(dm)$ vanishing on the Gleason part of m and let $A_m = \left\{f \in A: \int f dm = 0\right\}$.

THEOREM. *Let M be a closed invariant subspace of $L^2(dm)$ such that the linear span of $A_m M$ is dense in M but the subspace $R = \{f \in M: f \perp I^\infty M\}$ is nontrivial and has the same support set E as M. Then M has the form $\chi_E \cdot F \cdot (\bar{I}^2)^\perp$ for some unimodular function F.*

A modified form of the result holds for $1 \leqq p \leqq \infty$. This theorem is applied to give a complete characterization of the invariant subspaces of $L^p(dm)$ when A is the standard algebra on the torus associated with a lexicographic ordering of the dual group and m is normalized Haar measure.

1. Invariant subspaces. In 1949 Beurling [1], using function analytic methods, showed that all the closed invariant subspaces of H^2 of the circle have the form $M = FH^2$, where $|F| \equiv 1$ a.e. In 1958 Helson and Lowdenslager [3] and [4] extended the result to some but not all subspaces of the H^2 space of the torus, using Hilbert space methods. In the past 10 years the latter arguments have been extended by Hoffman [5, Th. 5.5, p. 293], Srinivasan [8], [9], and others to prove the following generalized Beurling theorem. If m is a representing measure for a logmodular algebra A and if M is an invariant subspace of $L^2(dm)$ which is *simply* invariant, i.e., if

(1) the linear span of $A_m M$ is not dense in M,

then $M = FH^2$ for $|F| \equiv 1$. In the general case (even the torus case) not all invariant subspaces satisfy this hypothesis. Our purpose is to extend the characterization by weakening hypothesis (1).

We assume throughout the paper that $A = A(X)$ is a logmodular algebra [5] of continuous complex-valued functions on a compact Hausdorff space X and that m is the unique representing measure on X for a complex homomorphism of A, i.e., $\int fgdm = \int fdm \int gdm$ for all $f, g \in A$. Furthermore we assume that this complex homomorphism lies in a Gleason part $P(m)$ containing more than one element. A function $f \in H^\infty(dm)$ is called *inner* if $|f| \equiv 1$. For each $f \in H^2(dm)$ we write $\hat{f}(\varphi) = \int fd\varphi$ for φ in $P(m)$, where φ also denotes the representing measure for the homomorphism φ.

In [10] Wermer showed (for A a Dirichlet algebra) that there exists an inner function Z such that $\hat{Z}$ maps $P(m)$ onto $\{\lambda: |\lambda| < 1\}$ and such that the equation

(2) $G(\hat{Z}(\varphi)) = \hat{f}(\varphi)$

associates with each f in $H^2(dm)$ an analytic function $G(\lambda) = \sum_{n=0}^\infty a_n\lambda^n$ for $|\lambda| < 1$ where $a_n = \int \bar{Z}^n fdm$. (See [5] for the extension to logmodular algebras.) Denote by F the boundary value function of G (i.e., the function in $L^2(d\theta)$ whose Fourier coefficients are a_n, where $d\theta$ is normalized Lebesgue measure on $\{|\lambda| = 1\}$).

Elementary arguments (including the Riesz-Fischer theorem) establish that the mapping $\Phi(f) = F$ can be extended to a bounded linear transformation of $L^2(dm)$ onto $L^2(d\theta)$, using the fact that $L^2(dm) = H^2(dm) \oplus \bar{H}^2_m(dm)$ [5, Th. 5.4, p. 293].

Denote by $\mathscr{Z}^p$ the closure (in $L^p(dm)$) of the polynomials in Z; denote by $\mathscr{L}^p$ the closure (in $L^p(dm)$) of the polynomials in Z and $\bar{Z}$. (For $p = \infty$, the closure is taken in the w^* topology.) Thus $\mathscr{L}^2 = \overline{\mathscr{Z}^2} \oplus \mathscr{Z}^2_m$ and Φ, restricted to $\mathscr{L}^2$, is an isometric isomorphism onto $L^2(d\theta)$, induced by the correspondence $Z \rightarrow e^{i\theta}$.

Actually Φ can be extended to a continuous transformation of $L^1(dm)$ onto $L^1(d\theta)$ induced by formula (2) and for $1 \leqq p \leqq \infty$ carrying $\mathscr{L}^p$ isometrically onto $L^p(d\theta)$. (This map also carries $H^p(dm)$ onto $H^p(d\theta)$.) This follows from the following result of Lumer [6, Th. 3, p. 285] (and our Lemma 5 below): The correspondence $Z \rightarrow e^{i\theta}$ induces an isometric isomorphism of $\mathscr{L}^p$ onto $L^p(d\theta)$ for each $p, 1 \leqq P \leqq \infty$, which carries $\mathscr{L}^p$ onto $H^p(d\theta)$. See also Merrill [7, Proof of Th. 1]. For f and $g \in L^2(dm)$, $\Phi(fg) = \Phi(f)\Phi(g)$ (see the proof of Lemma 10 in Wermer [10]). We call Φ the natural homomorphism of $L^1(dm)$ onto $L^1(d\theta)$.

Define $I^p = \left\{f \in H^p(dm): \int \bar{Z}^n fdm = 0, n = 0, 1, 2, \cdots\right\}$ for $1 \leqq p \leqq \infty$, so that $H^2(dm) = \mathscr{Z}^2 \oplus I^2$. Using (2) it is not hard to check that $I^p = \{f \in H^p(dm): \hat{f}(\varphi) = 0, \varphi \in P(m)\}$. For any subset $S \subseteqq L^2(dm)$, denote by $[S]$ the closed linear span of S.

DEFINITION. Let M be a closed invariant subspace of $L^p(dm)$. M

is called *simply invariant* if $A_m M$ is not dense in M (w^* dense for $p=\infty$) and doubly invariant if $\bar{A}M \subseteqq M$. We call M *sesqui-invariant* if $\bar{Z}M \subseteqq M$ but M is not invariant under $\bar{A}$.

There exist closed invariant subspaces of $L^2(dm)$ which are sesqui-invariant, i.e., neither simply nor doubly invariant. For example, let $M = I^2$. If I^2 satisfied (1) so that it had the form FH^2, F inner, then F would be in I^2, so that $\bar{Z}F$ would be in I^2 by Lemma 1 below. But if $I^2 = FH^2$, then $\bar{Z} \in H^2$, which is not the case.

Our main purpose in § 2 is to relax hypothesis (1) and to obtain a characterization of certain invariant subspaces of $L^2(dm)$ not covered by the Beurling theorem, in terms of the support set of M, a unimodular function, and I^2. At the end we extend the result to $1 \leqq p \leqq \infty$. Examples in which I^2 is nontrivial are given in § 3 together with applications of the main theorem. First we give three lemmas of a preliminary nature which collect elementary and known facts.

LEMMA 1. *If $f \in I^2$, then $\bar{Z}^n f \in I^2$.*

Proof. Clearly it suffices to show that $\bar{Z}f \in H^2$, for then $\bar{Z}f \perp \mathscr{Z}^2$ and hence $\bar{Z}f \in I^2$. Let $h \in H^2_m(dm)$ and write

$$a_n = \int \bar{Z}^n f dm,\ b_n = \int \bar{Z}^n h dm\ .$$

Then $\int \bar{Z} f h dm = a_0 b_1 + a_1 b_0 = 0$ so $\bar{Z}f \in H^2$.

LEMMA 2. *Let $M \subseteqq L^2(dm)$ be a closed subspace. Then the following are equivalent*

(i) $AM \subseteqq M$

(ii) $H^\infty M \subseteqq M$

(iii) $H^\infty_m M = ZM = [A_m M]$.

Proof. That (i) implies (ii) follows from the w^* density of A in $H^\infty(dm)$. To see that (ii) implies (iii) observe that by definition of Z, $H^2_m = ZH^2$ and hence $H^1_m = ZH^1$, by taking closure in L^1. By considering conjugate spaces and applying Corollary to Theorem 6.1 in Hoffman [5, p. 298], we have $H^\infty_m = ZH^\infty$. Using (ii), $H^\infty_m M = ZH^\infty M \subseteqq ZM \subseteqq H^\infty_m M$. In any case $H^\infty_m M = [A_m M]$ by the w^* density of A_m in H^∞_m. This establishes (iii).

To show that (iii) implies (i), it suffices to show (iii) implies (ii). We have seen that $H^\infty_m = ZH^\infty$ or $\bar{Z}H^\infty_m = H^\infty$. Using (iii) this yields $H^\infty M = \bar{Z}H^\infty_m M \subseteqq \bar{Z}ZM = M$.

LEMMA 3. *Let $M \subseteqq L^2(dm)$ be a closed invariant subspace. Then* *the following are equivalent.*

(a) $M = FH^2$ *for some unimodular function* F.

(b) $M \ominus [A_m M] \neq \{0\}$.

(c) $M \ominus ZM \neq \{0\}$.

(d) M *is not invariant under* $\bar{Z}$.

Proof. The equivalence of (a) and (b) is the generalized Beurling theorem. Items (b) and (c) are equivalent by Lemma 2. If (a) holds then so does (d). For if M were invariant under $\bar{Z}$ then since $F \in M$, $\bar{Z}F \in M = FH^2$, so that $\bar{Z} \in H^2$ which is not the case. On the other hand, if (d) holds, ZM is a proper closed subspace of M, i.e., (c) holds.

DEFINITION. If $f \in L^1(dm)$, we define the *support set* of f (denoted by E_f) as the complement of a set of maximal measure on which f is null. If M is a closed subspace of $L^1(dm)$, the support set of M (denoted by E_M) is defined as the complement of a set of maximal measure on which all $f \in M$ are null. Clearly E_f and E_M are defined only up to sets of measure zero.

2. The invariant subspace theorem.

THEOREM 1. *Let A be a logmodular algebra and m a fixed representing measure such that the part $P(m)$ contains more than one element. Let M be a closed sesqui-invariant subspace of $L^2(dm)$ and let E be the support set of M. Let $R = M \ominus [I^\infty M]$ and $L = M^\perp \ominus [\bar{I}^\infty M^\perp]$ where $M^\perp = \{f \in \chi_E L^2(dm): f \perp M\}$. Then*

(3) *L is nontrivial and the support set of L is E if and only if $\chi_E \in \mathscr{L}^2$ and M has the form $M = \chi_E \cdot F \cdot I^2$ for some unimodular function F, and*

(4) *R is nontrivial and the support set of R is E if and only if $\chi_E \in \mathscr{L}^2$ and M has the form $M = \chi_E \cdot F \cdot (\bar{I}^2)^\perp = \chi_E \cdot F \cdot (\mathscr{L}^2 \oplus I^2)$ for some unimodular function F.*

We need several lemmas, the key fact being Lemma 8.

LEMMA 4. *Let Z be the Wermer embedding function. If θ is Lebesgue measure on T, then $\theta\{Z(x): x \in X\} = 1$ and $m(Z^{-1}(E)) = 0$ if and only if $\theta(E) = 0$, for each measurable subset E of T. Moreover, if F in $L^1(d\theta)$ corresponds to $f \in \mathscr{L}^1$ under the natural homomorphism Φ, then $f(x) = F(Z(x))$ a.e.*

Proof. Suppose that $\theta(Z(X)) < 1$. Then there exists a closed set $K \subseteqq T \backslash Z(X)$ such that $\theta(K) > 0$. The functions $f_n(t) = 1/(1 + n\rho(t, K))$, where ρ denotes distance, are continuous for each n and converge to

$\chi_K(t)$ pointwise everywhere and in $L^2(d\theta)$. Let g_n and g denote the images in $\mathscr{L}^2$ of f_n and χ_K, respectively, under the natural correspondence. Hence $g_n \to g$ in $L^2(dm)$ and by passing to a subsequence we may assume that $g_n(x) \to g(x)$ a.e. (dm). Since the f_n may be approximated by trigonometric polynomials, $g_n(x) = f_n(Z(x))$ a.e. (dm), and the latter sequence converges to zero a.e. (dm) by the definition of the f_n. Hence $g(x) = 0$ a.e. (dm). But this contradicts the fact that g corresponds to a nonzero function. Thus $\theta(Z(X)) = 1$.

This also proves that if $\theta(E) > 0$, then $m(Z^{-1}(E)) > 0$. Now suppose that $\theta(E) = 0$, i.e., that $\chi_S(t) = 1$ a.e. $(d\theta)$, where $S = T\backslash E$. Choose closed sets $K_1 \subseteqq K_2 \subseteqq, \cdots, \subseteqq S$, such that $\theta(K_n) \to \theta(S)$. Using the argument of the previous paragraph, we can show that the characteristic function of K_n corresponds to that of $Z^{-1}(K_n)$. Thus the characteristic function of $Z^{-1}(K_n)$ converges in $L^2(dm)$ to the function 1. But the characteristic function of $Z^{-1}(K_n)$ also converges to that of $Z^{-1}(\cup K_n)$. Thus the latter function is 1 a.e. Thus $m(Z^{-1}(S)) = 1$ so that $m(Z^{-1}(E)) = 0$.

To obtain the last assertion of the lemma, let $F \in L^1(d\theta)$ and f the corresponding function in the isomorphic image of $L^1(d\theta)$ in $L^1(dm)$. Choose a sequence F_n of polynomials in $e^{i\theta}$ and $e^{-i\theta}$ which converge to F in $L^1(d\theta)$ and a.e. Let f_n correspond to F_n so that $f_n \to f$ in $L^1(dm)$ and can be replaced by a subsequence which converges a.e.

Since F_n are polynomials, $f_n(x) = F_n(Z(x))$ a.e. (dm). Since $F_n(t) \to F(t)$ a.e. $(d\theta)$, the first part of the lemma implies that $F_n(Z(x)) \to F(Z(x))$ a.e. (dm). Thus $f(x) = F(Z(x))$ a.e.

LEMMA 5. *If* $1 \leqq p \leqq \infty$, *then*

$$H^p(dm) = \mathscr{X}^p \oplus I^p$$

where $\oplus$ *denotes algebraic direct sum. Denote by* N^p *the closure of* $\bar{I}^p \oplus I^p$ *in* $L^p(dm)$ *(norm closure for* $1 \leqq p < \infty$; w^* *closure for* $p = \infty$). *Then*

$$L^p(dm) = \mathscr{L}^p \oplus N^p .$$

Proof. First assume $1 < p \leqq \infty$. If $f \in H^p(dm)$, then f defines a bounded linear functional on $L^q(dm)$ which (via Lumer's isometry) induces a bounded linear functional on $L^q(d\theta)$, which in turn is represented by some $F \in L^p(d\theta)$. It is easy to show that

$$\int Z^n f dm = \int e^{in\theta} F d\theta$$

for all integers n. Hence $F \in H^p(d\theta)$, and by Lumer's isometry there exists $g \in \mathscr{X}^p$ with

$$\int Z^n f dm = \int Z^n g dm$$

so that $f - g \in I^p$. Hence $H^p(dm) = \mathscr{X}^p \oplus I^p, 1 < p \leqq \infty$.

Now let $p = 1$ and $f \in H^1(dm)$. Since the lemma holds for $p = 2$ and H^1 is the closure of $\mathscr{X}^2 \oplus I^2$, there exists $g_n \in \mathscr{X}^2$ and $h_n \in I^2$ such that the functions $f_n = g_n + h_n$ converge in L^1 to f. We will have shown that $H^1(dm) = \mathscr{X}^1 \oplus I^1$ if we can establish that $\{g_n\}$ forms a Cauchy sequence. For this it suffices to show that whenever $f = g + h$ for $g \in \mathscr{X}^2$ and $h \in I^2$, then $\| g \|_1 \leqq \| f \|_1$.

Applying Lumer's isometry for $p = 1$ for the second equality and for $p = \infty$ for the fourth, we have

$$\begin{aligned} \| g \|_1 &= \int_X | g | dm = \int_T | \Phi(g) | d\theta = \sup_{\|q\|_\infty \leqq 1} \left| \int_T \Phi(g)\Phi(q) d\theta \right| \\ &= \sup_{\|q\|_\infty \leqq 1} \left| \int_X gq dm \right| = \sup_{\|q\|_\infty \leqq 1} \left| \int_X fq dm \right| \leqq \| f \|_1 , \end{aligned}$$

where q ranges over $\mathscr{L}^\infty$. Thus $H^p(dm) = \mathscr{X}^p \oplus I^p, 1 \leqq p \leqq \infty$.

For the second part of the lemma, denote

$$M^p = \left\{ f \in L^p(dm) : \int Z^n f dm = 0 \text{ all integers } n \right\} .$$

It can be shown that $L^p(dm) = \mathscr{L}^p \oplus M^p$ by the same arguments we used for the H^p case. We can complete the proof of the lemma by showing that $M^p = N^p, 1 \leqq p \leqq \infty$.

Clearly $N^p \subseteqq M^p$. Let $f \in M^p$. Since $\bar{H}^p_m(dm) \oplus H^p(dm)$ is dense in $L^p(dm)$ [5, Th. 6.7, p. 305] and $H^p(dm) = \mathscr{X}^p \oplus I^p$ by the first part of the lemma, we can choose $g_n \in \mathscr{L}^p$ and $h_n \in N^p$ such that

$$\int k(g_n + h_n) dm \longrightarrow \int kf dm$$

for all $k \in L^q(dm)$. Write $k = k_1 + k_2$ where $k_1 \in \mathscr{L}^q$ and $k_2 \in M^q$. Thus

$$\int k_1 g_n dm = \int k_1 (g_n + h_n) dm \longrightarrow \int k_1 f dm = 0 .$$

Also $\int k_2 g_n dm = 0$. Thus $\int k g_n dm \to 0$. Since the subspace N^p is norm closed for $1 \leqq p \leqq \infty$, it is also weakly closed, so $f \in N^p$. If $p = \infty$, clearly $f \in N^\infty$.

LEMMA 6. *Let M be a closed sesqui-invariant subspace of $L^2(dm)$, and let $R = M \ominus [I^\infty M]$. If $f \in R$ and E_f is the support set of f, write $\tilde{f}$ for the characteristic function of E_f. Then $\tilde{f} \perp I^2$.*

Proof. Observe that for any $f, g \in R$ the function $f\bar{g}$ is orthogonal to both I^∞ and $\bar{I}^\infty$. For if $h \in I^\infty$, $gh \in I^\infty M$ so that $f \perp gh$, i.e., $f\bar{g} \perp h$. Similarly $f\bar{g} \perp \bar{I}^\infty$. In particular $|f|^2 = f\bar{f} \perp I^\infty$ and $\bar{I}^\infty$. It follows easily from Lemma 5 that $|f|^2$ lies in $\mathscr{L}^1$. If F is the function in $L^1(d\theta)$ corresponding to $|f|^2$, we have $|f(x)|^2 = F(Z(x))$ by Lemma 4. In particular $f(x) = 0$ if and only if $F(Z(x)) = 0$ so that $\tilde{f} = \tilde{F} \circ Z$. Since $\tilde{F} \in L^2(d\theta)$, it follows that $\tilde{f} \in \mathscr{L}^2$, i.e., $\tilde{f} \perp I^2$.

LEMMA 7. *Suppose that M is a closed sesqui-invariant subspace of $L^2(dm)$ and let $R = M \ominus [I^\infty M]$. Then there exists $f \in R$ with $E_f = E_R$.*

Proof. If $f, g \in R$, note that there exists $h \in R$ with $E_h = E_f \cup E_g$. For let $F = E_g \backslash E_f$. Since $\chi_F \in \mathscr{L}^2$ by Lemma 6, $\chi_F g \in R$. Then $f + \chi_F g \in R$ and has support set $E_f \cup E_g$. Now let $\alpha = \sup\{m(E_f): f \in R\}$. Choose $f_n \in R$ with $m(E_{f_n}) \to \alpha$ and $E_{f_1} \subseteqq E_{f_2} \subseteqq \cdots$. Alter the functions f_n by the technique above so that their supports are disjoint. Then $f_0 = \sum_{n=1}^\infty 2^{-n} f_n \in R$ and has support G with $m(G) = \alpha$. If $m(E_R) > \alpha$, then there would exist a set of positive measure in $E_R \backslash G$ and a function $g \in R$ such that g would not vanish on that set. But then $E_{f_0} \cup E_g$ is the support set for some function in R, although $m(E_{f_0} \cup E_g) > \alpha$. This contradiction shows that $E_{f_0} = E_R$.

LEMMA 8. *Let M be a closed sesqui-invariant subspace of $L^2(dm)$, $R = M \ominus [I^\infty M]$, and let E be the support set of R. Then there exists a unimodular function $F \in L^2(dm)$ such that $\chi_E F \in R$. If $m(E) = 1$, then $F \in R$.*

Proof. By Lemma 7, there exists $f \in R$ with $E_f = E$. Define

$$F(x) = \begin{cases} f(x)/|f(x)|, & x \in E \\ 1, & x \notin E. \end{cases}$$

Then $|F(x)| = 1$ a.e., and $f = F|f|$.

As in the proof of Lemma 6, since $f \in R$, there exists a function $F \in L^1(d\theta)$ such that $|f(x)|^2 = F(Z(x))$ a.e. Thus $F \geqq 0$ a.e. and $\sqrt{F} \in L^2(d\theta)$. Let h be the function in the isomorphic image of $L^2(d\theta)$ corresponding to $\sqrt{F}$. By Lemma 4, $\sqrt{F}(Z(x)) = h(x)$ a.e., i.e., $|f| = h \in \mathscr{L}^2$. It follows that $f = F|f| \in F\mathscr{L}^2$. Clearly $[Z^n f] \subseteqq F\mathscr{L}^2$ for all integers n. Writing $N = [Z^n f]$, we have $\bar{F}N = [Z^n \bar{F} f]$. But $Z^n \bar{F} f = Z^n(|f|/f)f = Z^n|f|$ on E, and is zero off E. Therefore $Z^n \bar{F} f \in \mathscr{L}^2$, so that $\bar{F}N \subseteqq \mathscr{L}^2$. However, $\bar{F}N$ is invariant under Z and $\bar{Z}$, so that its isomorphic image in $L^2(d\theta)$ is doubly invariant and must have the form $QL^2(d\theta)$ where $Q = Q^2 \in L^2(d\theta)$. Thus $\bar{F}N = q\mathscr{L}^2$ where q is the corresponding idempotent in $\mathscr{L}^2$. It is clear from the definition of N that $q = \chi_E$. Hence $N = F\chi_E\mathscr{L}^2$, so that $F\chi_E \in N \subseteqq R$.

REMARK. If M is a closed sesqui-invariant subspace of $L^2(dm)$, then $M^\perp$ (as defined earlier) is a closed subspace of $L^2(dm)$ invariant under $\bar{H}^\infty(dm)$ and Z. Let $L = M^\perp \ominus [\bar{I}^\infty M^\perp]$. Then dual forms of Lemma 6, 7, and 8 hold with L in place of R.

Proof of Theorem 1. First we assume that $M = \chi_E F I^2$ for some unimodular function F and that $\chi_E \in \mathscr{L}^2$ and show that $\chi_E F \in L$, so that $E_L = E$. To this end let $h \in I^2$. Then

$$\int \chi_E \bar{F} \chi_E F h dm = \int \chi_E h dm = 0$$

by assumption, so that $\chi_E F \in M^\perp$. To see that $\chi_E F \perp \bar{I}^\infty M^\perp$, let $h \in I^\infty$ and $k \in M^\perp$. It suffices to show that $\chi_E F \perp \bar{h}k$, i.e., that $\chi_E F h \perp k$. But this follows since $k \perp M$. A dual argument shows that $M = \chi_E F(\bar{I}^2)^\perp$ and $\chi_E \in \mathscr{L}^2$ imply that $\chi_E F \in R$ so that $E_R = E$.

Conversely, let us suppose that $E_L = E$. By Lemma 8, there exists a unimodular function $F \in L^2(dm)$ such that $\chi_E F \in L$. It follows that

$$FH^2(dm) \supseteqq M \supseteqq \chi_E F I^2 . \tag{5}$$

To prove the first inclusion in (5) it suffices to show that $M^\perp \supseteqq F\bar{H}^2_m$ where this time $M^\perp$ denotes the orthogonal complement in *all* of $L^2(dm)$. Thus let $h \in A_m$, so that $hM \subseteqq M$ and $\chi_E F \perp hM$. Since the functions in M vanish off E by assumption it follows that $F \perp hM$, i.e., $F\bar{h} \perp M$, so that $F\bar{H}^2_m \subseteqq M^\perp$ as required.

To obtain the second inclusion, let $g \in I^\infty$ and suppose that $f \perp M$ in $\chi_E L^2(dm)$. It follows easily from Lemma 5 that I^∞ is dense in I^2. Thus it suffices to show that $\chi_E F g \perp f$, i.e., that $\chi_E F \perp \bar{g} f$. But this follows since $\chi_E F \perp \bar{I}^\infty M^\perp$ by construction.

Multiplying (5) by $\bar{F}$ we have

$$H^2(dm) \supseteqq \bar{F} M \supseteqq \chi_E I^2 . \tag{6}$$

We use the invariance of M under $\bar{Z}$ to show that $\bar{F}M = \chi_E I^2$. For let $f \in \bar{F}M$ and write $f = f_1 + f_2$ where $f_1 \in \mathscr{Z}^2$, $f_2 \in I^2$. By Lemma 6, $\chi_E \in \mathscr{L}^2$ so that

$$f = \chi_E f = \chi_E f_1 + \chi_E f_2$$

is the unique orthogonal decomposition of f into $\mathscr{L}^2$ and I^2. However, since f and $\chi_E f_2$ are both in H^2 (Lemma 1), it follows that $\chi_E f_1 \in H^2$. Therefore $\chi_E f_1 \in \mathscr{Z}^2$. But $\chi_E f_1$ vanishes on the complement of E so that either (i) $m(E) = 1$, or (ii) $\chi_E f_1 \equiv 0$.

If case (i) holds, $H^2 \supseteqq \bar{F}M \supseteqq I^2$ so that either $\bar{F}M = I^2$ or there exists $f \in \bar{F}M$ with $\int \bar{Z}^n f dm \neq 0$ for some nonnegative integer n. By considering the least integer for which such an f exists, it is not hard to see that $\bar{F}M$ would not be invariant under $\bar{Z}$. Thus $M = FI^2$.

If case (ii) holds, $f = \chi_E f_2 \in I^2$ and $\chi_E f = f \in \chi_E I^2$. Thus $\bar{F}M \subseteqq \chi_E I^2$. Together with (6) this implies that $\bar{F}M = \chi_E I^2$. So that $M = \chi_E \cdot F \cdot I^2$.

We turn now to case (4) in which R is nontrivial and the support of $(R) = E$. Let $N = M^\perp = \{f \in L^2(dm)\colon E_f \subseteqq E \text{ and } f \perp M\}$. Then N is the complex conjugate of a sesqui-invariant subspace and

$$N^\perp \ominus [I^\infty N^\perp] = M \ominus [I^\infty M] = R \,.$$

We apply (a trivial modification of) the first part of the theorem to N. For this we need to know that $E_N = E$. If $G = E \backslash E_N$ is not the null set, then $\chi_G \cdot L^2(dm) \subseteqq M$ which is not possible. Thus $E_N = E$ and $N = \chi_E \cdot F \cdot \bar{I}^2$ for some unimodular function F. Hence

$$M = N^\perp = \chi_E F \cdot (\bar{I}^2)^\perp = \chi_E \cdot F \cdot (\mathscr{L}^2 \oplus I^2) \,.$$

We now extend the main result to a more general class of subspaces of $L^2(dm)$.

THEOREM 2. *Let M be a closed sesqui-invariant subspace of $L^2(dm)$. Let $M_1 = \{f \in M\colon f \cdot L^\infty(dm) \subseteqq M\}$ and $M_2 = M \ominus M_1$, and $R_2 = M_2 \ominus [I^\infty M_2]$. Assume that E_2, the support set of M_2 is the same as the support set of R_2. Then*

$$M = \chi_{E_1} \cdot L^2(dm) \oplus \chi_{E_2} \cdot F \cdot I^2$$

where F is unimodular, E_1 is the support set of M_1 and $\chi_{E_2} \perp I^2$.

Proof. Since M_1 is a closed doubly invariant subspace of $L^2(dm)$, there exists a measurable set $E_1 \subseteqq X$ such that $M_1 = \chi_{E_1} \cdot L^2(dm)$ (see Helson [2, Th. 2, p. 7]). It is easy to check that

$$M_2 = \{f \in M\colon f \equiv 0 \quad \text{on} \quad E_1\} \,.$$

Since M is sesqui-invariant, $M_2 \neq \{0\}$, and is itself sesqui-invariant. By Theorem 1, $M_2 = \chi_{E_2} \cdot F \cdot I^2$ for some $\chi_{E_2} \perp I^2$ and F unimodular.

The final theorem of this section characterizes the invariant subspaces of $L^p(dm)$ for $1 \leqq p \leqq \infty$.

THEOREM 3. *Fix p in the range $1 \leqq p \leqq \infty$. Let M be a closed sesqui-invariant subspace of $L^p(dm)$ and let E be the support set of M. Let $R = \{f \in M \cap L^q\colon f \perp I^\infty M\}$ and $L = \{f \in M^\perp \cap L^p\colon f \perp \bar{I}^\infty M^\perp\}$ where q is the conjugate index to p and $M^\perp = \{f \in \chi_E \cdot L^q(dm)\colon f \perp M\}$.*

Then

(i) $M = \chi_E \cdot F(\mathscr{L}^p + I^p)$ *where* $\chi_E \in \mathscr{L}^2$ *and* F *is a unimodular function if and only if* E *is the support set for* R.

(ii) $M = \chi_E \cdot F \cdot I^p$ *where* $\chi_E \in \mathscr{L}^2$ *and* F *is a unimodular function if and only if* E *is the support set for* L.

Proof. It is easy to show that if M has form (i) or (ii) then E is the support set of R or L, respectively. Let us prove the converse. First we prove the theorem for $p = 1$. Suppose that E is the support set of R. Let $N = M \cap L^2(dm)$; N is a closed sesqui-invariant subspace of $L^2(dm)$. Let $R^* = \{f \in N : f \perp I^\infty N\}$. Since $R \subset R^*$, we get E is the support set of R^* which in turn is the support set for N. Applying the L^2 invariant subspace theorem to N, we get $N = \chi_E \cdot F(\mathscr{L}^2 + I^2)$. Since $N \subseteqq M$, we get $\chi_E \cdot F(\mathscr{L}^1 + I^1) \subseteqq M$. For $f \in M$, define $k = |f|^{1/2}$ for $|f| \geqq 1$ and 1 for $|f| < 1$. Take $h \in H^2(dm)$ outer such that $|h| = k$. It is easy to see that $1/h \in H^\infty(dm)$ and therefore $f/h \in M$. Since $f/h \in L^2(dm)$ also, we get $f/h \in N = \chi_E \cdot F(\mathscr{L}^2 + I^2)$ and therefore $f \in \chi_E \cdot F(\mathscr{L}^1 + I^1)$. Thus we get $M = \chi_E \cdot F \cdot (\mathscr{L}^1 + I^1)$. When E is the support set for L, we get $M = \chi_E \cdot F \cdot I^1$ by applying an argument similar to the above.

Now let us prove the theorem for $p = \infty$. Suppose that E is the support set for R. Let $N = [M]$ (where [ ] denotes closure in $L^2(dm)$. Let $R^* = \{f \in N : f \perp I^\infty N\}$. It is clear that E is the support set for N which in turn is the support set for R^*. By the L^2 invariant subspace theorem we get $N = \chi_E \cdot F(\mathscr{L}^2 + I^2)$. Since $M \subseteqq N \cap L^\infty(dm)$, we get $M \subseteqq \chi_E \cdot F(\mathscr{L}^\infty + I^\infty)$. By applying the L^1 invariant subspace theorem to $\overline{M^\perp}$, we get $M^\perp = \chi_E \cdot \bar{G} \cdot \overline{I^1}$, $|G| \equiv 1$. It is easy to see that $\chi_E \cdot \overline{GI^1} \perp \chi_E F(\mathscr{L}^\infty + I^\infty)$ and therefore $M = \chi_E \cdot F(\mathscr{L}^\infty + I^\infty)$. When E is the support set for L, we get $M = \chi_E \cdot F \cdot I^\infty$ by applying an argument similar to the above. The proof for $1 < p < 2$ is similar to the one for $p = 1$ and that for $2 < p < \infty$ is similar to the one for $p = \infty$. Thus the theorem is true for $1 \leqq p \leqq \infty$.

3. **Applications.** We give an example of a logmodular algebra and a representing measure m for which I^2 is nontrivial and show that the above theorems, together with known results, completely characterize the invariant subspaces of $L^2(dm)$.

EXAMPLE 1. Let $T = \{\lambda \in C : |\lambda| = 1\}$ and let $A = A(T^2)$ be the logmodular algebra of continuous functions on T^2 which are uniform limits of polynomials in $e^{in\theta}e^{im\varphi}$ where

$$(n, m) \in S = \{(n, m) : n > 0\} \cup \{(0, m) : m \geqq 0\} .$$

The maximal ideal space of A can be identified with

$$(\{\theta: |\theta| \leqq 1\} \times T) \cup (\{0\} \times \{\varphi: |\varphi| \leqq 1\}) ,$$

with normalized Haar measure m identified with $\theta = \varphi = 0$. The part of m is $\{0\} \times \{\varphi: |\varphi| < 1\}$. The Wermer embedding function is given by $Z = e^{i\varphi}$, $\mathscr{Z}^2$ is the L^2 closure of the polynomials in $e^{im\varphi}$, $m = 0, 1, \cdots$, and I^2 is the L^2 closure of the polynomials in $e^{in\theta}e^{im\varphi}$ for $n \geqq 1$.

Let now M be a closed invariant subspace of $L^2(dm)$. Observe that M is doubly invariant if and only if $e^{i\theta}M = M$. In this case $M = \chi_E \cdot L^2(dm)$, for some measurable set $E \subseteqq T^2$.

If $M \ominus e^{i\theta}M \neq \{0\}$ and $M = e^{i\varphi}M$ we show that $R \neq \{0\}$ and that $E_R = E_2$ (see Theorem 2). To see that $R = M \ominus e^{i\theta}M$, let $g \in M$, $g \perp e^{i\theta}M$. Since M is sesqui-invariant $g \perp e^{-im\varphi}e^{i\theta}M$, for $m = 1, 2, \cdots$. Hence $g \perp [I^\infty M]$.

Define $M_1 = \{f \in M: e^{-in\theta}f \in M, n = 1, 2, \cdots\}$ and $M_2 = M \ominus M_1$. Then $M_1 = \chi_{E_1} \cdot L^2(dm)$ for some measurable E_1. We show that Theorem 2 applies to M_2. Let K be the complement of E_R in T^2.

Since $\chi_K \in \mathscr{L}^2$, we get $\chi_K M_2 \subseteqq M_2$. Also $\chi_K \cdot M_2 \perp R$ so $\chi_K M_2 \subset e^{i\theta}M_2$ and therefore $\chi_K M_2 = \chi_K(e^{i\theta}M_2)$. But M_2 cannot contain a doubly invariant subspace, so $E_R = E_2$. Theorem 2 applies and

$$M_2 = \chi_{E_2} \cdot F'(\bar{I}^2)^\perp$$

for some unimodular function F'. Writing $F = e^{-i\theta}F'$, we have $M_2 = \chi_{E_2} \cdot F \cdot I^2$. Note that the proofs of Lemmas 4, 6, and 7 are much simpler for the torus case than for the general case.

If $M \ominus e^{i\varphi}M \neq \{0\}$, then $M = FH^2$ by the generalized Beurling theorem.

Suppose that we now replace $T \times T$ with $B \times T$, where B is the Bohr compactification of the real line and consider $A = A(B \times T)$. Again Haar measure is associated with a nontrivial part. Denote by $\chi_\tau(x)$ the characters on B, where $\tau \in R$. I^2 is generated by the characters $\chi_\tau(x)e^{im\varphi}$ for $\tau > 0$. Clearly (3) holds for $M = \chi_\tau I^2$ and (4) holds for $M = \chi_\tau(I^2 \oplus \mathscr{L}^2)$, for any fixed τ. However one can use the example in Helson and Lowdenslager [4] to construct a sesqui-invariant subspace of $H^2(dm)$ for which both L and R are trivial.

REFERENCES

1. A. Beurling, *On the problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), 239–255.

2. H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

3. H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. **99** (1958), 165–202.

4. H. Helson and D. Lowdenslager, *Invariant subspaces*, Proc. Internat. Symp. Linear Spaces, Jerusalem, 1960, 251–262. Macmillan (Pergamon), New York, 1961.

5. K. Hoffman, *Analytic functions and logmodular Banach algebras*, Acta Math. **108** (1962), 271–317.

6. G. Lumer, *H^∞ and the imbedding of the classical H^p spaces in arbitrary ones*, Function algebras (Proc. Internat. Symp. on Function algebras, Tulane Univ., 1965), F. Birtel, Ed., Scott Foresman, Chicago, 1966, 285-286.
7. S. Merrill, *Maximality of certain algebras $H^\infty(dm)$*, Math. Zeit. **106** (1968), 261-266.
8. T. P. Srinivasan, *Simply invariant subspaces*, Bull. Amer. Math. Soc. **69** (1963), 706-709.
9. ———, *Simply invariant subspaces and generalized analytic functions*, Proc. Amer. Math. Soc. (1965), 813-818.
10. J. Wermer, *Dirichlet algebras*, Duke Math. J. **27** (1960), 373-382.

Received October 4, 1968. The first named author was partially supported by NSF grant GP 7548 at the University of Rochester.

THE UNIVERSITY OF ROCHESTER
ROCHESTER, NEW YORK