# A RADICAL COINCIDING WITH THE LOWER RADICAL IN ASSOCIATIVE AND ALTERNATIVE RINGS

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In a recent paper by the second author a construction was given which was shown to coincide with the lower radical in all associative rings. In the present paper this construction is considered in various classes of not necessarily associative rings. It is shown that while the construction still defines a radical, it will in general properly contain the lower radical. More precisely, it is shown that the radical constructed coincides with the lower radical if the semisimple class of the lower radical is hereditary (or, equivalently, if the radical of a ring always contains the radicals of all its ideals).

From this condition it follows that the construction coincides with the lower radical in all associative and alternative rings, but an example is given which shows that this is not true in general. We conclude by showing that an apparently quite different construction due to J. F. Watters [5] yields exactly the same class of rings.

We will assume that all rings considered in this paper are from some universal class $\mathcal{U}$ of not necessarily associative rings. We will use the following construction, which is equivalent to that of [4]. Let $\mathcal{A}$ be an arbitrary class of rings and $\mathcal{A}_0$ its homomorphic closure. Then define $\mathcal{A}_n = \{R \in \mathcal{U} \mid R$ has a nonzero ideal $I \in \mathcal{A}_{n-1}\}$, and $A_\omega = \bigcup_n \mathcal{A}_n$. Then define $\mathcal{Y}(\mathcal{A}) = \{R \in \mathcal{U} \mid R/I \in \mathcal{A}_\omega$ for all ideals $I$ of $R\}$. It is clear from this definition that we have

LEMMA 1. $\mathcal{A} \subseteq \mathcal{A}_0 \subseteq \mathcal{Y}(\mathcal{A})$.

LEMMA 2. $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{Y}(\mathcal{A}) \subseteq \mathcal{Y}(\mathcal{B})$.

It is also easy to check that the proof of [4, Th. 1] makes no use of associativity. Thus we may state

THEOREM 1. $\mathcal{Y}(\mathcal{A})$ is a radical class.

We will replace [4, Th. 2] by the following generalization:

THEOREM 2. If $\mathcal{P}$ is a radical sub-class of $\mathcal{U}$, then $\mathcal{P} = \mathcal{Y}(\mathcal{P})$ if either of the following two equivalent conditions is satisfied:
  ( i )  The semisimple class $\mathcal{SP}$ of $\mathcal{P}$ is hereditary,
  (ii)  Writing $\mathcal{P}(R)$ for the $\mathcal{P}$-radical of $R$, then $\mathcal{P}(I) \subseteq \mathcal{P}(R)$

*for every ideal I of every $R \in \mathscr{U}$.*

*Proof.* The equivalence of (i) and (ii) follows from [1, Lemma 2, p. 595]. Thus assume that $\mathscr{SP}$ is hereditary. By Lemma 1 we have $\mathscr{P} \subseteq \mathscr{U}(\mathscr{P})$ and suppose there could exist $R \in \mathscr{U}(\mathscr{P}), R \notin \mathscr{P}$. Then $R$ has a nonzero homomorphic image in $\mathscr{SP}$, so (without loss of generality) assume $R \in \mathscr{U}(\mathscr{P}) \cap \mathscr{SP}$. Thus $R \in \mathscr{P}_n$ for some $n$, and since $\mathscr{P} \cap \mathscr{SP} = 0$ it is clear that $n \geq 1$. Let $m$ be the smallest integer such that there exists a nonzero $R \in \mathscr{P}_m \cap \mathscr{SP}$. Then $R$ has a nonzero ideal $I \in \mathscr{P}_{m-1}$. Since $\mathscr{SP}$ is hereditary $I \in \mathscr{SP}$ contrary to the minimality of $m$. Thus $\mathscr{P} = \mathscr{U}(\mathscr{P})$.

COROLLARY 1. *If $\mathscr{P}$ is a radical class then in any associative or alternative ring the $\mathscr{P}$-radical and the $\mathscr{U}(\mathscr{P})$-radical coincide.*

*Proof.* This is clear since the intersection of $\mathscr{P}$ with any universal associative or alternative class is again a radical class and semisimple classes are always hereditary in associative [2, Corollary 2, p. 125] or alternative classes [1, Corollary 2, p. 602].

Note that a sufficient condition for property (ii) is that $\mathscr{P}(I)$ shall be an ideal of $R$. This is already known to be true in associative rings [2, Th. 47, p. 124] or alternative rings [1, Th. 2, p. 600]. From this last remark it also follows that the proof of [4, Th. 2] could have been applied equally well to alternative rings.

THEOREM 3. *Let $\mathscr{L}(\mathscr{A})$ be the lower radical for an arbitrary class $\mathscr{A}$. Then $\mathscr{L}(\mathscr{A}) = \mathscr{U}(\mathscr{A})$ if $\mathscr{SL}(\mathscr{A})$ is hereditary.*

*Proof.* Suppose $\mathscr{SL}(\mathscr{A})$ is hereditary. From Lemma 1 and the minimality of $\mathscr{L}(\mathscr{A})$ among radical classes containing $\mathscr{A}$ [2, Lemma 5, p. 13] it follows that $\mathscr{L}(\mathscr{A}) \subseteq \mathscr{U}(\mathscr{A})$. But by Lemma 2, $\mathscr{A} \subseteq \mathscr{L}(\mathscr{A})$ implies $\mathscr{U}(\mathscr{A}) \subseteq \mathscr{U}(\mathscr{L}(\mathscr{A}))$. Then if $\mathscr{SL}(\mathscr{A})$ is hereditary if follows from Theorem 2 that $\mathscr{U}(\mathscr{L}(\mathscr{A})) = \mathscr{L}(\mathscr{A})$ and so $\mathscr{L}(\mathscr{A}) = \mathscr{U}(\mathscr{A})$.

We can thus conclude that the $\mathscr{U}(\mathscr{A})$-radical coincides with the lower radical in any associative or alternative ring.

*Note.* The class $\mathscr{I}$ of all idempotent rings is a radical class whose semisimple class is nonhereditary [3, Th. 2, p. 1116]. It is also true that $\mathscr{U}(\mathscr{I}) = \mathscr{I}$ for if $R \notin \mathscr{I}$ then since all subrings of $R/R^2$ are zero rings, $R/R^2$ has no accessible subrings in $\mathscr{I}$. Thus $R \notin \mathscr{U}(\mathscr{I})$ and so $\mathscr{U}(\mathscr{I}) = \mathscr{I}$. This example shows that the conditions of Theorems 2 and 3 are not necessary.

Also remark that there are classes $\mathscr{A}$ for which $\mathscr{U}(\mathscr{A})$ is not the lower radical. One example is the class $\mathscr{B} = \mathscr{L}(\mathscr{X})$ where $\mathscr{X}$

is the class of all zero rings. Let $R$ be the ring constructed [see 3] over $Z_2$ in symbols $u, v, w$ satisfying relations $u^2 = w^2 = 0$, $uv = vu = uw = u$, and $wu = vw = wv = v^2 = v$. The only ideal of $R$ is $H = \{0, u, v, u + v\}$ for which $H^2 = H$. Now $\mathscr{F}$ is a hereditary class so by a result of A.E. Hoffman [see 6, Theorem 1,] we have $\mathscr{F} \cap \mathscr{L}(\mathscr{Z}) = 0$. Now the lower radical of a hereditary class is hereditary. Thus $\mathscr{L}(\mathscr{Z})$ is hereditary and so $R \notin \mathscr{L}(\mathscr{Z})$. On the other hand, $R/H \in \mathscr{Z}$ and $R$ has the accessible subring $J = \{0, u\} \in \mathscr{Z}$. Hence $R \in \mathscr{Y}(\mathscr{Z})$.

It should also be noted that while $\mathscr{Y}(\mathscr{A})$ need not equal $\mathscr{L}(\mathscr{A})$, it is nevertheless true for all classes $\mathscr{A}$ that $\mathscr{Y}(\mathscr{L}(\mathscr{A})) = \mathscr{Y}(\mathscr{A})$. This is an easy consequence of the fact that $\mathscr{Y}(\mathscr{Y}(\mathscr{A})) = \mathscr{Y}(\mathscr{A})$.

In a paper [5] which is soon to appear the following construction is given: Let $\mathscr{M}$ be an arbitrary homomorphically closed subclass of some universal class $\mathscr{U}$. For $R \in \mathscr{U}$ define $M_{\sigma 0} = 0$, and for an arbitrary ordinal $\alpha$ $M_{\sigma \alpha} = \bigcup_{\beta < \alpha} M_{\sigma \beta}$, if $\alpha$ is a limit ordinal, or $M_{\sigma \alpha}/M_{\sigma \beta}$ is the ideal of $R/M_{\sigma \beta}$ generated by all accessible $\mathscr{M}$-subrings of $R/M_{\sigma \beta}$, whenever $\alpha = \beta + 1$. If $\gamma$ is the ordinal for which $M_{\sigma \gamma} = M_{\sigma \gamma + 1}$, write $M_\sigma(R) = M_{\sigma \gamma}$ and let $\mathscr{M}'_\sigma = \{R \in \mathscr{U} \mid M_\sigma(R) = R\}$.

THEOREM 4. $\mathscr{M}'_\sigma = \mathscr{Y}(\mathscr{M})$.

*Proof.* Let $R$ be a ring for which $M_\sigma(R) = R$, and let $I \neq R$ be an ideal of $R$. Then $0 = M_{\sigma 0} \subseteq I$, and there must exist some ordinal $\alpha$ such that $M_{\sigma \alpha} \subseteq I$ but $M_{\sigma \alpha + 1} \nsubseteq I$. Write $A = M_{\sigma \alpha}$, $B = M_{\sigma \alpha + 1}$. Since $I$ is an ideal of $R$, it follows from the definition that $B/A$ contains an accessible $\mathscr{M}$-subring $W/A$ of $R/A$ such that $W \nsubseteq I$. Then the natural homomorphism $R/A \to R/I$ gives $W/A \to W'$ with $W'$ accessible. Thus since $\mathscr{M}$ is homomorphically closed, we have a nonzero $W' \in \mathscr{M}$. It follows that $R/I \in \mathscr{M}_n$ for some $n$ and since $I$ was arbitrary, $R \in \mathscr{Y}(\mathscr{M})$. Thus $\mathscr{M}'_\sigma \subseteq \mathscr{Y}(\mathscr{M})$.

The converse is clear, for suppose $R \in \mathscr{Y}(\mathscr{M})$ and $I = M_\sigma(R) = M_{\sigma \gamma}(R)$. If $I \neq R$, it follows that $R/I$ has an accessible $\mathscr{M}$-subring, contradicting $M_{\sigma \gamma}(R) = M_{\sigma \gamma + 1}(R)$. Thus $M_\sigma(R) = R$ whence $R \in \mathscr{M}_\sigma'$ and so $\mathscr{M}_\sigma' = \mathscr{Y}(\mathscr{M})$.

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