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A SUBCOLLECTION OF ALGEBRAS IN A COLLECTION OF BANACH SPACES

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Let $D(p, r)$ with $1 \leqq p < \infty$ and $-\infty < r < +\infty$ denote the Banach space consisting of certain analytic functions $f(z)$ defined in the unit disk. A function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a member of $D(p, r)$ if and only if

$$\sum_{n=0}^{\infty} (n+1)^r \mid a_n \mid^p < \infty .$$

We define the norm of f in $D(p, r)$ by

$$\| f \|_{p,r} = \left(\sum_{n=0}^{\infty} (n+1)^r \mid a_n \mid^p \right) 1/p .$$

By the product of two functions f and g in $D(p, r)$ we shall mean their product as functions, i.e., $[f. g](z) = f(z)g(z)$. The purpose of this paper is to discover which of the spaces $D(p, r)$ are algebras.

THEOREM 1. *If $D(p, r)$ is an algebra, then there exists a real $c > 0$ with $\| fg \| \leqq c \| f \| \| g \|$ for every $f, g \in D(p, r)$.*

Proof. Let h be a fixed element of $D(p, r)$. It suffices to show the map $f \rightarrow hf$ is a bounded linear transformation from $D(p, r)$ to itself. The proof is based on the closed graph theorem [2, p. 306]. Suppose h is a multiplier from $D(p_1, r_1)$ to $D(p_2, r_2)$ and suppose

(i) $f_n \rightarrow f$ in $D(p_1, r_1)$ and

(ii) $hf_n \rightarrow g$ in $D(p_2, r_2)$.

Then $f_n(z) \rightarrow f(z)$ for each z in the unit disk and so $h(z)f_n(z) \rightarrow h(z)f(z)$. On the other hand by (ii), $h(z)f_n(z) \rightarrow g(z)$ for each z in the unit disk. Hence $g = hf$, and so by the closed graph theorem multiplication by h is a continuous linear transformation. It follows from this [2, p. 183] that $D(p, r)$ is equivalent to a Banach algebra, and from this the theorem follows immediately.

COROLLARY 1. *If $D(p, r)$ is an algebra and $c > 0$ as above, then* $|f(z)| \leqq c \| f \| \ \forall f \in D(p, r)$ and $|z| < 1$.

Proof. For each f in $D(p, r)$ let T_f denote the multiplication operator from $D(p, r)$ to itself determined by f, i.e., $T_f(g) = fg$. Then for z_0 satisfying $|z_0| < 1$ the map $T_f \rightarrow f(z_0)$ is a multiplicative linear functional on the Banach algebra of multiplication operators

$$T_f, f \in D(p, r)$$

with the usual norm. Hence

$$|f(z_0)| \leqq \|T_f\| = \sup_{\|g\|=1} \|fg\| \leqq c\|f\|, g \in D(p, r) .$$

THEOREM 2. *If* $p = 1$, *then* $D(p, r)$ *is no algebra for* $r < 0$. *And if* $1 < p < \infty$, *then* $D(p, r)$ *is no algebra for* $r \leqq p - 1$.

Proof. The function $f(z) = \sum_{n=0}^{\infty} [1/(n+1)]z^n$ is an unbounded function on $|z| < 1$ but lies in $D(1, r)$ if $r \leqq 0$. And similarly the function $f(z) = \sum_{n=0}^{\infty} 1/[(n+1)\log(n+1)]z^n$ is an unbounded function on $|z| < 1$ in $D(p, r)$ if $p > 1$ and $r \leqq p - 1$. Therefore by Corollary 1 the spaces are not algebras.

THEOREM 3. *If* $p = 1$, *then* $D(p, r)$ *is an algebra for* $r \geqq 0$, *and if* $1 < p < \infty$ *then* $D(p, r)$ *is an algebra for* $r > p - 1$.

Proof. (i) Suppose first $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ lie in $D(1, r)$ with $r \geqq 0$. We will show $fg \in D(1, r)$

$$\begin{aligned}
\|fg\| &= \sum_{n=0}^{\infty} (n+1)^r \left|\sum_{k=0}^{n} a_k b_{n-k}\right| \leqq \sum_{n=0}^{\infty} (n+1)^r \sum_{k=0}^{n} |a_k|\,|b_{n-k}| \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} (n+1)^r |a_k|\,|b_{n-k}| \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} (j+k+1)^r |a_k|\,|b_j| \quad \text{where} \quad j = n - k \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} (j+k+1)^r/[(k+1)^r(j+1)^r](k+1)^r|a_k|(j+1)^r|b_j| \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} [(j+k+1)/(jk+j+k+1)]^r(k+1)^r|a_k|(j+1)^r|b_j| \\
&\leqq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} (k+1)^r|a_k|(j+1)^r|b_j| \\
&= \|f\|\,\|g\| .
\end{aligned}$$

(ii) Now suppose $r > p - 1$, and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

be two elements of $D(p, r)$. We will show there is a constant K such that $\|fg\| \leqq K\|f\|\,\|g\|$. Define q by the equation $1/p + 1/q = 1$.

$$\|fg\|^p = \sum_{n=0}^{\infty} (n+1)^r \left|\sum_{k=0}^{n} a_k b_{n-k}\right|^p \leqq \sum_{n=0}^{\infty} (n+1)^r$$

$$\left\{\sum_{k=0}^{n} 1/[(k+1)^{r/p}(n-k+1)^{r/p}](k+1)^{r/p}|a_k|(n-k+1)^{r/p}|b_{n-k}|\right\}^p .$$

Applying Holder's inequality we get

$$\begin{aligned}
&\|fg\|^p \\
&\leqq \sum_{n=0}^{\infty} (n+1)^r \Big\{\Big(\sum_{k=0}^{n} [1/\{(k+1)^{r/p}(n-k+1)^{r/p}\}]^q\Big)^{1/q} \\
&\qquad\qquad \Big(\sum_{k=0}^{n} (k+1)^r \,|\, a_k \,|^p\, (n-k+1)^r \,|\, b_{n-k} \,|^p\Big)^{1/p}\Big\}^p \\
&= \sum_{n=0}^{\infty} [C_n] \sum_{k=0}^{n} (k+1)^r \,|\, a_k \,|^p\, (n-k+1)^r \,|\, b_{n-k} \,|^p \\
&\leqq \sup_n\, [C_n]\, \|f\|^p\, \|g\|^p
\end{aligned}$$

where

$$C_n = (n+1)^r\Big(\sum_{k=0}^{n} \{1/[k+1)^{r/p}(n-k+1)^{r/p}]\}^q\Big)^{p/q} .$$

We complete the proof of the theorem by showing

$$\sup_n\, [C_n] < \infty .$$

$$\begin{aligned}
C_n &= (n+1)^r\Big(\sum_{k=0}^{n} \{1/[(k+1)^{r/p}(n-k+1)^{r/p}]\}^q\Big)^{p/q} \\
&= (n+1)^r\Big(\sum_{k=0}^{n} 1/(n+2)^{rq/p}\{1/(k+1) + 1/(n-k+1)\}^{rq/p}\Big)^{p/q} \\
&= [(n+1)/(n+2)]^r\Big[\sum_{k=0}^{n} \{1/(k+1) + 1/(n-k+1)\}^{rq/p}\Big]^{p/q} \\
&\leqq \Big[\sum_{k=0}^{n} \{2/(k+1)\}^{rq/p}\Big]^{p/q} \\
&\leqq 2^r\Big[\sum_{k=0}^{\infty} 1/(k+1)^{rq/p}\Big]^{p/q}
\end{aligned}$$

since

$$rq/p = r/(p-1) > 1 .$$

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