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TWO CHARACTERIZATIONS OF QUASI-FROBENIUS RINGS

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The purpose of this paper is to characterize quasi-Frobenius, QF, rings in terms of relationships assumed to exist for each cyclic or finitely generated left module between the module and its second dual, where duality is with respect to the ring. More specifically we prove that a left perfect ring is QF if every cyclic left module is reflexive or every finitely generated left module is (isomorphic to) a submodule of a free module. For rings with minimum condition on left or right ideals this later condition is equivalent to every finitely generated left module being torsionless or to the ring being a cogenerator in the category of finitely generated left modules. If annihilator relations are defined by means of the natural pairing between a module and its dual, this condition is also equivalent to every submodule of a finitely generated left module being an annulet.

One of Nakayama's [14] original characterizations of QF rings was as rings with minimum condition on left or right ideals in which every left ideal is a left annulet and every right ideal is a right annulet. Ikeda and Nakayama [7] proved that a finite dimensional algebra in which every left ideal is a left annulet is QF but Nakayama [15] gave an example which shows that this weaker assumption does not characterize QF rings even when one assumes minimum condition on both left and right ideals. For an arbitrary ring every left ideal being a left annulet is easily seen to be equivalent to every cyclic left module being torsionless. Thus our results may be regarded as an attempt to extend the one sided result of Ikeda and Nakayama mentioned above by making weaker finiteness assumptions and stronger assumptions about the relation of the members of various classes of left modules to their second duals. Our results are also related to those of Morita, Kawada and Tachikawa [12] who proved that a ring with minimum condition is QF if every left module is a submodule of a free module and of Faith and Walker [5] who recently showed that this characterization is still valid without the assumption of minimum condition. Finally we call attention to the investigations of B.L. Osofsky [17] which we cannot adequately relate to the above results in a few words.

1. Preliminaries. Let M and N be modules over a ring R. The bimodule character of R induces an R-module structure on $M^* =$

$\operatorname{Hom}_R(M, R)$, the *dual* of M, of the opposite hand from that of M and an R-homomorphism $\alpha: M \to N$ gives rise to an R-homomorphism $\alpha^*: N^* \to M^*$ so that $*$ is a contravariant functor. There is a natural homomorphism $\sigma_M: M \to M^{**}$ defined by

$$\sigma_M(x)(f) = f(x), \text{ for all } x \in M \text{ and } f \in M^* ,$$

which gives a natural transformation from the identity functor to the functor $**$ and M is called *torsionless* (resp. *reflexive*) of σ_M is a monomorphism (resp. isomorphism). A thorough discussion of these ideas and the basic properties of $*$ is contained in Chapter 5 of the monograph of Jans [9] or they are summarized more concisely in Part II of Bass' paper [1]. Other excellent references for this duality theory are Dieudonne [2] or Morita [11]. We shall draw freely upon the above sources and our notation and terminology will be as consistent as possible with that of Bass [1] and Jans [9]. In particular R will always denote a ring with identity, J the (Jacobson) radical of R and all R-modules will be unitary.

Let X be a subset of R. Then $l(X) = \{r \in R: rx = 0 \text{ for all } x \in X\}$ and $r(X) = \{r \in R: xr = 0 \text{ for all } x \in X\}$. Any left (resp. right) ideal of the form $l(X)$ (resp. $r(X)$) is a *left* (resp. *right*) *annulet*. R is *quasi-Frobenius*, *QF*, in case: (1) each right ideal is a right annulet; (2) each left ideal is a left annulet; and (3) R has the minimum condition on left or right ideals. *QF* rings have numerous other charaterizations to which we shall refer as needed.

A module $M \subseteqq N$ is small in N if $N = M + K$ implies $K = N$ for any submodule K of N. A projective module P is called a projective cover of a module L if there exists an epimorphism of P onto M whose kernel is small in P. *Semi-perfect* rings are those for which every finitely generated module has a projective cover. They are also characterized by the assumptions that R/J satisfies the minimum condition on one sided ideals and idempotents can be lifted modulo J (see [1]). R is called *left perfect* if every left R-module has a projective cover. The equivalence of the following conditions was established by Bass [1]: (1) R is left perfect, (2) R/J satisfies the minimum condition on one sided ideals and every nonzero right R-module contains a simple right R-module, (3) a direct limit of projective left R-modules is projective, and (4) R/J satisfies the minimum condition on one sided ideals and if $\{a_i: i = 0, 1, \cdots\} \subseteqq J$, there is an n such that $a_0 a_1 \cdots a_n = 0$. Thus if R satisfies the minimum condition on left or right ideals, R is left perfect.

2. Rings whose cyclic modules are reflexive. If C is a cyclic left (right) R-module, C is isomorphic to R/I where I is a left (right) ideal of R and C is torsionless if and only if I is a left (right) annulet.

A right (left) ideal I of R satisfies *condition* S if and only if every R-homomorphism of I into R is given by left (right) multiplication by an element of R. It is easy to verify that I satisfies condition S if and only if $\mathrm{Ext}^1_R(R/I, R) = 0$ (see [9]).

LEMMA 1. *Let I be a left ideal of R. Then R/I is reflexive if and only if*

(1) $l(r(I)) = I$ *and*

(2) $r(I)$ *satisfies condition* S.

Proof. $(R/I)^*$ is isomorphic to $r(I)$ under the map δ^{-1} defined by $\delta^{-1}(f) = f(1 + I)$ for all $f \in (R/I)^*$. It is clear that the map $\alpha: R/l(r(I) \rightarrow r(I)^*$ which sends each $r + l(r(I))$ into left multiplication by r is an R-monomorphism. Furthermore, α is an isomorphism if and only if $r(I)$ satisfies condition S. A straight forward verification shows that $\alpha \circ \eta = \delta^* \circ \sigma_{R/I}$, where η is the natural projection of R/I onto $R/l(r(I))$. If R/I is reflexive, (1) holds since η must be a monomorphism and (2) holds since α must be an epimorphism as δ^* and $\sigma_{R/I}$ are both isomorphisms. If (1) and (2) hold, η, α, and δ^* are all isomorphisms and hence so is $\sigma_{R/I}$.

The next result is immediate from Lemma 1 (and its right hand analog) and the "injective test theorem" [9, p. 49].

THEOREM 1. *Every cyclic left and every cyclic right R-module is reflexive if and only if ${}_RR$ and R_R are injective R-modules, every left ideal of R is a left annulet, and every right ideal of R is a right annulet.*

The above theorem is not new. It can be obtained by specializing results of Morita [11, Lemma 2.1 and Th. 2.4]. Such a ring need not be QF although it must be semi-perfect (see Osofsky [17, Example 1, p. 378 and Th. 2, p. 380]).

LEMMA 2. *Let R be a left perfect ring. If every cyclic left R-module is reflexive, then R_R is an injective right R-module.*

Proof. First we show that if S is any simple right R-module, S^* is simple or zero. Since $S \cong R/I$ for some maximal right ideal I, $S^* \cong l(I)$. But $l(I) = 0$ or is a minimal left ideal of R. Otherwise, there is a left ideal $0 \subsetneqq L \subsetneqq l(I)$. Then $I \subseteqq r(l(I)) \subseteqq r(L) \subseteqq R$ and hence either $r(l(I)) = r(L)$or $r(L) = R$. But L is an annulet so either $L = l(r(L)) = l(r(l(I))) = l(I)$ or $L = l(r(L)) = l(R) = 0$. Since neither equality is possible we have reached a contradiction.

Let $C_1, \cdots, C_m$ be a complete set of nonisomorphic simple left R-modules. Since R is left perfect, each C_i^* contains a simple (right) submodule S_i. The inclusion map j of S_i into C_i^* induces a map $j^*: C_i^{**} \to S_i^*$ which is nontrivial since C_i^* is torsionless and is; therefore, an isomorphism since both S_i^* and $C_i^{**} \cong C_i$ are simple (left) R-modules. Hence $S_1, \cdots, S_m$ are distinct and so are a complete set of nonisomorphic simple right R-modules. Furthermore, they are torsionless and hence so is R_R/J since it is completely reducible. Thus J is a right annulet.

Now ${}_RR/l(J)$ is reflexive by assumption and hence Lemma 1 implies that $r(l(J)) = J$ satisfies condition S or what is the same thing that $\mathrm{Ext}_R^1(R_R/J, R_R) = 0$. Thus R_R is an injective right R-module (see [10, Prop. 2.6, p. 251]).

THEOREM 2. *The following statements are equivalent for any ring R.*

(a) *R is quasi-Frobenius.*

(b) *R is left perfect and every cyclic left R-module is reflexive.*

Proof. That (a) implies (b) is well known (see [2] or [8]). Since QF rings are right self injective (see [3, Th. 18, pp. 11–12]) this implication is also immediate from Lemma 1. Assume (b). By Lemma 2, R_R is an injective right R-module and so [3, Th. 18, pp. 11–12] will imply that R is QF if we show that R has the minimum condition on left ideals. In view of [17, Lemma 11, p. 382], it suffices for this to show that ${}_RJ/J^2$ is finitely generated. Suppose it is not. Then there exists an exact sequence

$$(\sharp) \qquad 0 \to \bigoplus_{i=1}^{\infty} S_i \to {}_RR/J^2 \to {}_RM \to 0 ,$$

where the S_i are isomorphic simple modules and ${}_RR/J^2$ and ${}_RM$ are reflexive. Dualizing $\sharp$ gives an exact sequence $0 \to {}_RM^* \to ({}_RR/J^2)^* \to \prod_{i=1}^{\infty} S_i^*$. Thus we have an exact sequence

$$(\sharp\sharp) \qquad 0 \to {}_RM^* \to ({}_RR/J^2)^* \to \bigoplus_{a \in A} S_a^* \to 0 ,$$

with each $S_a^* \cong S_i^*$, since $\prod_{i=1}^{\infty} S_i^*$ is completely reducible as each S_i^* is simple (see the proof of Lemma 2). Since R_R is an injective right R-module dualizing $\sharp\sharp$ we obtain the exact sequence

$$(\sharp\sharp\sharp) \qquad 0 \to \prod_{a \in A} S_A^{**} \to ({}_RR/J^2)^{**} \to {}_RM^{**} \to 0 .$$

Connecting $\sharp$ and $\sharp\sharp\sharp$ by means of the natural isomorphisms $\sigma_{R/J}$ and σ_M one verifies easily that there is induced an R-isomorphism between $\bigoplus_{i=1}^{\infty} S_i$ and $\prod_{a \in A} S_a^{**} \cong \prod_{a \in A} S_a$, with each $S_a \cong S_1$.

Since the hypotheses are preserved under passage to the basic ring (see [6] or [16]), we may assume that R is basic. Then R/J is a ring direct sum of division rings so that S_1 is a one dimensional vector space over a division ring D. Thus $\prod_{\alpha \in A} S_\alpha$ and $\bigoplus_{i=i}^{\infty} S_i$ are isomorphic both as R-modules and as D-vector spaces. However; since A is clearly infinite, D-dimension of $\prod_{\alpha \in A} S_\alpha$ = cardinality $(D^A) > \aleph_0$ = D-dimension of $\bigoplus_{i=1}^{\infty} S_i$. This contradiction completes the proof.

COROLLARY 1. *Let R be a ring with minimum condition on left or right ideals. If C is isomorphic to C^{**} for each cyclic left R-module, R is quasi-Frobenius.*

Proof. Since duals are torsionless (see [9]), every left ideal of R is a left annulet. Hence in either case R has minimum condition of left ideals. Since $C \cong C^{**}$ they have the same composition length and σ_C being a monomorphism must be an isomorphism.

3. Torsionless modules and submodules of free modules. If S is a set, we shall denote the cardinality of S by $|S|$ and if α is an ordinal number, we shall denote the cardinality of any set of order type α by $|\alpha|$. If M is an R-module, $\|M\|$ will denote the smallest cardinal number for which M has a set of R-generators of this cardinal. If $\{M_\alpha, \prod_\alpha^\beta\}_{\alpha,\beta \in D}$ is a directed system of R-modules and R-homomorphisms over a directed set D then we shall then denote the direct limit of this system by $\varinjlim_D M_\alpha$ (see [4] or [19]).

THEOREM 3. *The following statements are equivalent.*

(a) *R is a quasi-Frobenius ring.*

(b) *R is a left perfect ring and every finitely generated left R-module is (isomorphic to) a submodule of a projective left R-module.*

Proof. That (a) implies (b) is well known (see [5] or [12]). Assume (b). Then by [5, Corollary 5.6, p. 216] it suffices to show that every left R-module is a submodule of a projective R-module. Suppose this is not so. Then there exists a smallest cardinal τ for which there is a left R-module M with $\|M\| = \tau$ which cannot be embedded in a projective module. Then τ must be an infinite cardinal. Let G be a set of R-generators for M with $|G| = \|M\| = \tau$. Let γ be the smallest ordinal associated with the cardinal τ. Since τ is an infinite cardinal γ is a limit ordinal, i.e., γ does not have an immediate predecessor. Furthermore, if α is any ordinal less than γ, then $|\alpha| < |\gamma|$. The set of all ordinals strictly less than γ is a set of order type γ and cardinality $|\gamma| = \tau$. Let $\alpha \rightarrow g_\alpha$ be a one-to-one correspondence between the set of all ordinals $< \gamma$ and G. For each ordinal $\alpha < \gamma$ define M_α to be the submodule of M generated by $G_\alpha = \{g_\beta: \beta < \alpha\}$. Then $\{M_\alpha\}_{\alpha<\gamma}$ is a chain of submodules of M and $M = \bigcup_{\alpha<\gamma} M_\alpha$. Thus we may regard the M_α and their inclusion maps as a direct system having M as direct limit. The proof will be completed by showing the existence of a direct system $\{P_\alpha, \prod_\alpha^\beta\}_{\alpha,\beta<\gamma}$ of projective modules and a morphism ϕ of direct systems such that

(1) for all $\alpha < \gamma$, $\phi_\alpha: M_\alpha \to P_\alpha$ is a monomorphism and

(2) for each $\alpha < \gamma$, if $|\alpha|$ is finite, $||P_\alpha||$ is finite and if $|\alpha|$ is infinite, $||P_\alpha|| \leqq |\alpha|$.

For then we would have $\bar{\phi} = \underrightarrow{\lim}\, \phi_\alpha: \underrightarrow{\lim}\, M_\alpha = M \to \underrightarrow{\lim}\, P_\alpha = \bar{P}$. This is a contradiction since $\bar{P}$ is projective as R is left perfect and $\bar{\phi}$ is a monomorphism since each ϕ_α is monic. Condition 2 is used only in proving the existence of the desired direct system.

We prove the existence of this direct system and of ϕ by transfinite induction. We take $P_0 = 0$ and $\phi^0 = 0$. Now to show that if ϕ^β and P_β have been defined for all $0 \leqq \beta < \alpha < \gamma$, then ϕ^α and P_α can be defined. We consider two cases.

Case I (α not a limit ordinal). Define X to be the submodule

$$\{(\phi^{\alpha-1}(m), -m): m \in M_{\alpha-1}\}$$

of $P_{\alpha-1} \oplus M_\alpha$ and $T = (P_{\alpha-1} \oplus M_\alpha)/X$ and let f_1 and f_2 be the R-homomorphisms from $P_{\alpha-1}$ and M_α, respectively, into T obtained by composing the usual injections into the direct sum with the natural projection onto the quotient module. It is straightforward to verify that X was defined so that f_1 and f_2 are both monomorphisms and $f_2 = f_1 \circ \phi^{\alpha-1}$. Also $||T|| \leqq ||P_{\alpha-1}|| + ||M_\alpha|| \leqq ||P_{\alpha-1}|| + |\alpha|$ which, in view of Condition 2, is finite if $|\alpha|$ is finite and $\leqq |\alpha - 1| + |\alpha| = |\alpha| < |\gamma| = \tau$ if $|\alpha|$ is infinite. Thus by the choice of τ there exists a projective module P_α containing T and satisfying the requirement of condition (2). We can take $\phi^\alpha = i \circ f_2$, where, i is the inclusion map of T into P_α, $\prod_{\alpha-1}^\alpha = i \circ f_1$ and $\prod_\beta^\alpha = \prod_{\alpha-1}^\alpha \circ \prod_\beta^{\alpha-1}$ if $\beta < \alpha$.

Case II (α a limit ordinal). Since $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, it is the direct limit of these modules. Thus we may take $P_\alpha = \underrightarrow{\lim}_{\beta<\alpha} P_\beta$, $\phi^\alpha = \underrightarrow{\lim}_{\beta<\alpha} \phi^\beta$, and $\prod_\beta^\alpha$ to be the natural projection into the direct limit. It remains only to observe that $||P_\alpha|| \leqq |\alpha|$. But for each $\beta < \alpha$, P_β has a set of generators B_β with $|B_\beta| \leqq |\alpha|$ and so P_α is generated by $B = \bigcup_{\beta<\alpha} \prod_\beta^\alpha (B_\beta)$ and $|B| \leqq |\alpha||\alpha| = |\alpha|$.

Nakayama]15, p. 48] showed how to construct a ring with precisely three left ideals $R \supsetneqq J \supsetneqq 0$ which also has minimum condition on right ideals but isn't QF. There also exists a ring with the same three left ideals which doesn't even have the minimum condition on right ideals (see [18]). For either of these rings R, O, and $J = R/J$ are the only cyclic left modules so both of these rings have the property that every cyclic left R-module is contained in a free R-module, in fact, is contained in R. On the other hand Faith and Walker [5, Corollary 5.10, p. 217] have shown that if every cyclic left and every cyclic right R-module is contained in a projective R-module, R is QF.

COROLLARY 2. *Let R be a ring with minimum condition on left (or right) ideals. Then the following statements are equivalent.*

(a) *R is quasi-Frobenius.*

(b) *${}_RR$ is a cogenerator for the category of finitely generated left R-modules.*

(c) *Every finitely generated left R-module is torsionless.*

Proof. That (a) implies (b) is well known (see [5, Corollary 5.4, p. 216]) and the equivalence of (b) and (c) is immediate from the definitions. Assume (c) and let ${}_RM$ be finitely generated over R. Then $0 = \cap \ker f, f \in M^*$. Thus there exist $f_1, \cdots, f_n \in M^*$ such that $\bigcap_{i=1}^n \ker f_i = 0$. Hence there is a monomorphism of M into the free module $\bigoplus_{i=1}^n R_i$, where $R_i \cong R$ for $i = 1, \cdots, n$, defined by $m \to (f_1(m), \cdots, f_n(m))$. Thus Theorem 3 implies R is QF.

Osofsky [17, Example 1, p. 378] showed that even if R^R is an injective cogenerator R need not be QF but it is an open question whether or not a left perfect ring R such that R^R is an injective cogenerator is QF.

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