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ON UNIVERSAL TREE-LIKE CONTINUA

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R. M. Schori has conjectured that if T is a tree, but not an arc, then there is no universal T-like continuum. We show that if G is a finite collection of trees and there is a universal G-like continuum, then each element of G is an arc. It then follows that if G is a finite collection of one-dimensional (connected) polyhedra, and there is a universal G-like continuum, then each element of G is an arc.

1. Definitions. By a *continuum* here we mean a compact connected metric space; by a *polyhedron*, a nondegenerate (finitely) triangulable continuum. In a metric space, the distance between two points, A and B, is denoted by $d(A, B)$, and a similar notation is used for distances between points and point sets. The closure of a point set K is denoted by $\bar{K}$.

The point P of the continuum M is a *junction point* of M if and only if $M - P$ has at least three components.

A *tree* is a polyhedron that contains no simple closed curve. The point P of the tree T is an *endpoint* of T if and only if P is a noncutpoint of T.

The continuum M is an *n-od* if and only if n is a positive integer greater than 2 and there is a point P such that M is the sum of n arcs, each two intersecting only at P, which is an endpoint of both of them. If PQ is one of the n arcs, then $PQ - P$ is called a *ray* of M.

If $\varepsilon > 0$, a transformation f from a metric space X onto a space Y is called an *ε-map* if and only if f is continuous and if P is a point of Y, then $f^{-1}(P)$ has diameter $< \varepsilon$. The space X is *Y-like* if and only if there is an ε-map from X onto Y for each $\varepsilon > 0$. If G is a collection of spaces, the metric space X is *G-like* if and only if for each $\varepsilon > 0$, there is an ε-map from X onto some element of G [1].

2. Lemmas.

LEMMA 1. *If P is a junction point of the subcontinuum M of the continuum U, then there is an open set R in U containing P such that if R' is an open subset of R containing P, then there is a positive number ε such that every ε-map f from U onto a tree, T, throws some point of R' onto a junction point of T.*

Proof. Since $M - P$ has at least three components, $M - P$ is the sum of three mutually separated point sets, K_1, K_2, and K_3. For each $i \leqq 3$, let P_i denote a point of K_i. Let R denote an open set in U that contains P but not P_1, P_2, or P_3, and suppose R' is any open subset of R that contains P. Let ε denote a positive number less than the distance between any two of the sets $K_i - K_i \cdot R'$ $(i \leqq 3)$, and also less than $d(P_i, K_j)$, for $i \leqq 3, j \leqq 3, i \neq j$.

Now, suppose f is an ε-map from U onto a tree T. Since, if $i \leqq 3$, $\bar{K}_i$ is a continuum, $f(\bar{K}_i)$ contains an arc α_i from $f(P_i)$ to $f(P)$. If no two of these arcs intersect except at $f(P)$, then $f(P)$ is a junction point of T. If the arc α_1 intersects the arc α_2 in a point distinct from $f(P)$, let Q denote the first point of α_2 on α_1 from $f(P_1)$ to $f(P)$. Clearly, Q must also be the first point of α_1 on α_2 from $f(P_2)$ to $f(P)$. Hence the three arcs, $[f(P), Q]$ and $[Q, f(P_1)]$ on α_1, and $[Q, f(P_2)]$ on α_2, intersect only in the point Q, and Q is a junction point of T. Moreover, Q is a point of $f(R')$, since $f^{-1}(Q)$ intersects both K_1 and K_2, but cannot intersect both $K_1 - K_1 \cdot R'$ and $K_2 - K_2 \cdot R'$.

A similar argument suffices in case some other pair of the arcs α_1, α_2, and α_3 intersect in a point distinct from $f(P)$.

LEMMA 2. *If N is an n-od with junction point P, lying in a continuum U, there is a positive number ε such that if f is an ε-map from U onto a tree T with at most one junction point then* (1) *T is a j-od with junction point Q, and $j \geqq n$,* (2) *each endpoint of N is thrown by f into some ray of T, but no two into the same ray, and* (3) *if E is an endpoint of N and $f(P)$ lies in the ray of T that contains $f(E)$, then $f(P)$ lies in the arc in T from Q to $f(E)$.*

Proof. By Lemma 1 there is an open set R in U containing P and a positive number ε' such that (1) $\bar{R}$ contains no endpoint of N and (2) if f is an ε'-map from U onto a tree T_0, then $f(R)$ contains a junction point of T_0. Let $P_1, \cdots, P_n$ denote the endpoints of N and, for each $i \leqq n$, let Z_i denote the ray of N that contains P_i. Let ε denote a positive number less than each of the numbers ε', $d(P_i, R)$, and $d(P_i, N - Z_i)$, for $i \leqq n$, and suppose that f is an ε-map from U onto a tree T with at most one junction point.

Since f is also an ε'-map from U onto T, $f(R)$ contains a junction point Q of T. Hence T is, for some positive integer j, a j-od. Now, if $i \leqq n$, $d(P_1, R) > \varepsilon$ and Q is in $f(R)$, so $f(P_i) \neq Q$, and $f(P_i)$ lies in a ray of T.

Suppose i and k are two integers such that $f(P_i)$ and $f(P_k)$ lie in the same ray of T. The arc in T from $f(P_i)$ to $f(P_k)$ must contain $f(P)$, for otherwise either $f(Z_i)$ contains $f(P_k)$ or $f(Z_k)$ contains $f(P_i)$, neither of which is possible, since $d(P_i, N - Z_i) > \varepsilon$ and $d(P_k, N - Z_k) > \varepsilon$. But then if $m \leqq n$ and $i \neq m \neq k$, either (1) $f(P_m)$ lies in $f(Z_i + Z_k)$ or (2) $f(P_i + P_k)$ intersects $f(Z_m)$, neither of which is possible. So the images of different endpoints of N lie in different rays of T, and $j \geqq n$.

Finally, suppose $i \leqq n$ and $f(P)$ lies in the ray W of T that contains $f(P_i)$, but $f(P)$ is not on the arc in T from Q to $f(P_i)$. Then $f(P_i)$ is on the arc in T from Q to $f(P)$. So, if $k \leqq n$, and $k \neq i$, then since $f(P_k)$ is not in W, $f(Z_k)$ contains $f(P_i)$. But $d(P_i, N - Z_i) > \varepsilon$.

LEMMA 3. *Suppose* (1) I_1; I_2; *and* I_3 *are the intervals in the plane with endpoints* $(-1, 1)$, $(-1, -1)$; $(-1, 0)$, $(1, 0)$; *and* $(1, 1)$, $(1, -1)$, *respectively, and* (2) $H = I_1 + I_2 + I_3$. *Then if* T *is any tree with at least two junction points, and* $\varepsilon > 0$, *there is an* ε*-map from* H *onto* T.

Proof. Let A and B denote the points $(-1, 0)$ and $(1, 0)$, respectively. Since T has two junction points, T contains an arc α whose endpoints, X and Y, are junction points of T, but no other point of α is a junction point of T. Let E denote the sum of all the components of $T - X$ that do not contain $\alpha - X$. Then E contains two mutually exclusive arcs β_1 and β_2 such that if $i \leqq 2$, then β_i contains no junction point of T, and one endpoint of β_i is an endpoint of T. If $i \leqq 2$, let Q_i denote the endpoint of β_i that is not an endpoint of T. Then $[E - (\beta_1 + \beta_2)] + X + Q_1 + Q_2$ is a tree.

Now, suppose $\varepsilon > 0$. Let C_1; D; and C_2 denote the subintervals of I_1 with endpoints $(-1, 1)$, $(-1, \varepsilon/2)$; $(-1, \varepsilon/2)$, $(-1, -\varepsilon/2)$; and $(-1, -\varepsilon/2)$, $(-1, -1)$, respectively. There is a continuous transformation g_1 from I_1 onto $E + X$ such that (1) if $i \leqq 2$, $g_1 | C_i$ is a homeomorphism from C_i onto β_i, (2) $f(A) = X$, and (3) $f(D) = [E - (\beta_1 + \beta_2)] + X + Q_1 + Q_2$. Clearly, g_1 is an ε-map. Similarly, there is an ε-map from I_3 onto $[T - (E + \alpha)] + B$ which may be combined with a homeomorphism from I_2 onto α to obtain an ε-map from H onto T.

3. Theorems.

THEOREM 1. *If* k *is a positive integer and* G *is a collection each element of which is a tree with not more than* k *junction points, but some element of* G *has two junction points, then there is no universal* G*-like continuum.*

Proof. Suppose U is a universal G-like continuum. Then by Lemma 3, the continuum H defined in Lemma 3 is G-like, and so U contains a continuum H' homeomorphic to H. Let T denote an element of G such that no element of G has more junction points than T, and let j denote the number of junction points of T. Let T_0 denote the continuum obtained from T by replacing, with a pseudo-arc, each arc in T which is maximal with respect to the property that each interior point of it is of order 2, in such a way that T_0 is T-like, and hence G-like. Again, U contains a continuum T' homeomorphic to T_0.

Suppose that one of the junction points of H' is not also a junction point of T'. Then U contains at least $j+1$ points $P_1, P_2, \cdots P_{j+1}$ each of which is a junction point of a subcontinuum of U. By successive applications of Lemma 1, there is a positive number ε and a sequence $R_1, R_2, \cdots R_{j+1}$ of open sets in U such that (1) $d(R_i, R_n) > \varepsilon$, for $i \leqq j+1, n < j+1$, and $i \neq n$, and (2) if f is an ε-map from U onto a tree, T, then if $i \leqq j+1, f(R_i)$ contains a junction point, J_i, of T. Note that the points $J_1, J_2, \cdots, J_{j+1}$ must all be distinct; hence T must have at least $j+1$ junction points. But since U is G-like, U can be ε-mapped onto some tree in G, and no tree in G has $j+1$ junction points. Thus we have a contradiction, and both junction points, A and B, of H' are also junction points of T'.

So U contains both an arc from A to B, and a continuum (T') that contains A and B, but no arc from A to B. Since U is treelike, and so hereditarily unicoherent, this is impossible.

Thus, there is no universal G-like continuum.

THEOREM 2. *If G is a finite collection each element of which is a tree, and there is a universal G-like continuum, then each element of G is an arc.*

Proof. Suppose some element of G is not an arc, but U is a universal G-like continuum. If some element of G has two junction points, then Theorem 1 is contradicted. Thus each element of G is an arc or, for some n, an n-od. Let n denote the greatest positive integer j such that G contains a j-od. Then U contains (1) an n-od N, and (2) a continum H which is the sum of n pseudo-arcs, all joined at only one point. By arguments used in the proof of Theorem 1, the junction point, P, of N is also the junction point of H.

Let (1) ε_1 denote a positive integer for the subcontinuum N of U as in Lemma 2, (2) ε_2 and R denote a positive number and an open set in U, respectively, such that R contains P, and if E is an endpoint of N, then $d(E, R) > \varepsilon$, and (3) C denote the component of $U \cdot R$ that contains P.

$\bar{C}$ is a subset of N, for suppose A is a point of $\bar{C}$ not in N. Let ε denote a positive number less than ε_1, ε_2, and $d(A, N)$. Since U is G-like, there is an ε-map f from U onto an element T of G. Since $\varepsilon < \varepsilon_1$ we have, using Lemma 2, that (1) T is an n-od with junction point Q, (2) each ray of T contains the image of one, and only one, endpoint of N, and (3) there is an endpoint E of N such that $f(P)$ lies in the arc in T from Q to $f(E)$. Since $d(A, N) > \varepsilon$, $f(A)$ does not intersect $f(N)$, so there is an endpoint E' of N such that $f(E')$

lies in the arc in T from Q to $f(A)$. Since $\bar{C}$ is a continuum that contains A and a point of $f^{-1}(Q)$, $f(\bar{C})$ contains $f(A)$ and Q, and so $f(\bar{C})$ contains $f(E')$. But since $d(E', C) > \varepsilon$, this is impossible.

Thus $\bar{C}$ is a subset of N. Since the component C' of $H \cdot R$ that contains P is a subset of C, $\bar{C}'$ contains an arc. But H itself contains no arc, and we have a contradiction.

THEOREM 3. *If G is a finite collection each element of which is a one-dimensional polyhedron, and there is a universal G-like continuum, then each element of G is an arc.*

Proof. If some element of G contains a simple closed curve, then by a theorem of M.C. McCord [2, Th. 4, p. 72], there is no universal G-like continuum. So each element of G is a tree, and by Theorem 2, each element of G is an arc.

We note that if each element of G is an arc, there is a universal G-like continuum [3].

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