PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 3, 1969

POLYHEDRON INEQUALITY AND STRICT CONVEXITY

B. B. PHADKE

This paper considers convexity of functions defined on the "Grassmann cone" of simple r-vectors. It is proved that the strict polyhedron inequality does not imply strict convexity.

H. Busemann, in conjunction with others, (see [3]), has considered the problem of giving a suitable definition of the convexity of functions defined on nonconvex sets. An examination of various methods of defining convexity on the "Grassmann cone" (see [1]) is found in [2]. The most important open problems (see [3]) are whether weak convexity implies the area minimizing property (also called the polyhedron inequality) and whether the latter implies convexity. A modest result in this direction is proved below, namely, the strict area minimizing property does not imply strict convexity.

2. **Basic definitions.** Let a continuous function $\mathscr{F}$ be defined on the Grassmann cone G_r^n of the simple r-vectors R in the linear space V_r^n of all r-vectors $\tilde{R}$ (over the reals). Let $\mathscr{F}$ be positive homogeneous, i.e., $\mathscr{F}(\lambda R) = \lambda \mathscr{F}(R)$ for $\lambda \geqq 0$. To a Borel set F in an oriented r-flat $\mathscr{R}^+$ in the n-dimensional affine space A^n, we associate a simple r-vector as follows: $R = 0$ if F has r-dimensional measure 0, and otherwise $R = v_1 \wedge v_2 \wedge \cdots \wedge v_r$, is parallel to $\mathscr{R}^+$ and the measure of the parallelepiped spanned by $v_1, v_2, \cdots, v_r$ equals the measure of F. (Note a set of measure 0 and equality of measures in parallel r-flats are affine concepts and hence welldefined.) We denote below by $\mathscr{R}$ an r-flat parallel to an r-vector R passing through the origin.

DEFINITION 1. We say that $\mathscr{F}$ has the strict area minimizing property (*SFMA*) if: Whenever $R_0, R_1, \cdots, R_p$ are associated to r-dimensional faces of an r-dimensional oriented closed polyhedron P we have $\mathscr{F}(-R_0) < \Sigma \mathscr{F}(R_i)$, with $i = 1$ to p, unless $R_i = \lambda_i R_0$, $\lambda_i \geqq 0$ for all $i = 1$ to p (called the strict Polyhedron Inequality).

DEFINITION 2. $\mathscr{F}$ is said to be strictly weakly convex (*SWC*) if: Whenever R, R_1 and R_2 are simple, $R = R_1 + R_2, R_1$ is not a scalar multiple of R_2, we have $\mathscr{F}(R) < \mathscr{F}(R_1) + \mathscr{F}(R_2)$.

DEFINITION 3. $\mathscr{F}$ is said to be convex (C) if there exists a convex extension of $\mathscr{F}$ to V_r^n.

DEFINITION 4. $\mathscr{F}$ is said to be strictly convex (SC) if $\mathscr{F}$ is C and if there is at least one convex extension F of $\mathscr{F}$ to V_r^n which satisfies the following property: Whenever $\tilde{R} = \Sigma \tilde{R}_i$ with $\tilde{R}, \tilde{R}_i \in V_r^n$, $\tilde{R}$ is not a scalar multiple of all $\tilde{R}_i$, then $F(\tilde{R}) < \Sigma F(\tilde{R}_i)$.

In terms of these definitions we wish to prove below that: if $\mathscr{F}$ is SWC and C then it has the $SFMA$ and that if $\mathscr{F}$ is SWC and C it still need not be SC. This implies that the property $SFMA$ is weaker than the property SC.

3. Some algebraic facts. We collect below some algebraic facts whicn are either known or are relatively easy to prove.

(a) Let R_1 and R_2 be simple vectors. Then $R_1 + R_2$ is simple if and only if $\mathscr{R}_1$ and $\mathscr{R}_2$ intersect in a flat of dimension $\geqq r - 1$.

(b) Identify r-vectors with points representing them in V_r^n considered as an affine space. If a line in V_r^n contains three points corresponding to simple vectors, then the entire line consists of simple vectors. Put differently, if R_1 and R_2 are simple and $R_i + R_2$ is not simple, then the line joining R_1 and R_2 in V_r^n does not contain any simple vector other than R_1 and R_2.

Suppose next that R_1, R_2 and R_3 are simple but that $R_1 + R_j$ is nonsimple for all $i, j = 1$ to 3 when $i \neq j$. Then we have the following:

(c) The set $\{R_1, R_2, R_3\}$ is a linearly independent set of vectors.

(d) The plane π containing $\Delta R_1 R_2 R_3$ does not contain any line of simple vectors.

(e) The flat Ω spanned by the origin, R_1, R_2 and R_3 does not contain a 2-plane of simple vectors.

(f) If a line l lies in Ω and does not pass through the origin, then l cannot be a line of simple vectors, i.e., l cannot contain three distinct points corresponding to simple vectors.

4. An example. Busemann and Straus [2] give the following concrete example which we use here to illustrate the above algebraic facts. Let the vectors e_1, e_2, e_3, e_4 form a base for the four dimensional affine space A^4. Denote by e_{ij} the 2-vectors $e_i \wedge e_j$. Let Ω denote the flat spanned by the origin, e_{12}, e_{34} and $(e_1 + e_3) \wedge (e_2 + e_4)$ in V_2^4. We denote the vectors spanning Ω by z, R_1, R_2 and R_3 respectively. Then $R_i + R_j$ is nonsimple for all $i, j = 1$ to 3 when $i \neq j$. Thus any line l in Ω which does not pass through the origin cannot contain three distinct points representing simple vectors.

5. SWC with C is stronger than the $SFMA$.

LEMMA A. *If a function $\mathscr{F}$ is SWC and C then it has the SFMA.*

Proof. Let $R_0, R_1, R_2, \cdots, R_p$ be r-vectors corresponding to r-faces of an r-dimensional oriented closed polyhedron P. We need consider only the case when not all R_i are scalar multiples of R_0, $i > 0$. In such a case, since P is closed, some other faces which are not parallel to the face represented by R_0 intersect the face represented by R_0 in an $(r-1)$-dimensional set. Let R_1 be associated with one such face. Then from § 3a the vector $R_0 + R_1$ is simple. Also since P is closed we have $-(R_0 + R_1) = \sum_{i=2}^{p} R_i$. Thus $\sum_{i=2}^{p} R_i$ is also simple. But then the equation: $-R_0 = R_1 + \sum_{i=2}^{p} R_i$ shows that

$$\mathscr{F}(-R_0) < \mathscr{F}(R_1) + \mathscr{F}\left(\sum_{i=2}^{p} R_i\right),$$

and, since $\mathscr{F}$ is convex, $\mathscr{F}(-R_0) < \sum_{i=1}^{p} \mathscr{F}(R_i)$ so that $\mathscr{F}$ has the *SFMA*.

6. Existence of functions which are *SWC* and *C* but not *SC*.

LEMMA B. *There exist functions which are SWC and C but not SC.*

Proof. We actually construct an absolutely homogeneous function of this type. Take three simple unit vectors R_1, R_2, R_3 in V_r^n such that $R_i + R_j$ is nonsimple for all $i, j = 1$ to 3 with $i \neq j$. Choose unit vectors $\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_p$ in V_r^n such that the set of vectors $\{R_i \tilde{S}_j\}$, $i = 1$ to 3, $j = 1$ to p where

$$p = \binom{n}{r} - 3,$$

is a base for V_r^n. Thus given $\tilde{R} \in V_r^n$ we find unique numbers $\{a_i, b_j\}$ such that $\tilde{R} = \Sigma a_i R_i + \Sigma b_j \tilde{S}_j$. We denote this last written equality by the notation $\tilde{R} = (a_i, b_j)$. Now define the function $\mathscr{F}$ in V_r^n in the following manner:

If $\tilde{R} = (a_i, b_j)$ then $\mathscr{F}(\tilde{R}) = \sum_{i,j} (a_i^2 + b_j^2)^{1/2} + (\sum_j b_j^2)^{1/2}$ with $i = 1$ to 3 and $j = 1$ to p.

We verify that $\mathscr{F}$ has the required property.

(i) $\mathscr{F}$ is clearly absolutely homogeneous ($\mathscr{F}(\lambda R) = |\lambda| \mathscr{F}(R)$) and a convex function on V_r^n, hence convex on G_r^n.

(ii) We next show $\mathscr{F}$ is *SWC*. Let $R = (a_i, b_j)$ and

$$R' = (a_i', b_j')$$

be two simple r-vectors such that $R + R'$ is also simple. Assume further that $\mathscr{F}(R + R') = \mathscr{F}(R) + \mathscr{F}(R')$. We prove $\mathscr{R}$ is parallel to $\mathscr{R}'$. Assume that $\mathscr{R}$ is not parallel to $\mathscr{R}'$. Then the line l in V_r^n joining R to R' is a line of simple vectors and l does not pass through the origin. Therefore from the algebraic facts, l does not lie in the flat Ω spanned by the origin, R_1, R_2 and R_3. Therefore either a b_j or a b'_j is different from zero. Without loss of generality assume that $b_1 \neq 0$. We make the simple observation that when numbers α_i and β_i are such that $\alpha_i \leqq \beta_i$ and $\Sigma\alpha_i = \Sigma\beta_i$ then each $\alpha_k = \beta_k$. From this and $\mathscr{F}(R + R') = \mathscr{F}(R) + \mathscr{F}(R')$ we have the following equalities:

$$(\mathrm{E}) \qquad (\Sigma b_j^2)^{1/2} + (\Sigma b_j'^2)^{1/2} = (\Sigma(b_j + b'_j)^2)^{1/2} .$$

For all (i, j),

$$(\mathrm{E}_{ij}) \qquad (a_i^2 + b_j^2)^{1/2} + (a_i'^2 + b_j'^2)^{1/2} = ((a_i + a'_i)^2 + (b_j + b'_j)^2)^{1/2} .$$

From the equality (E) we see that there exists a number μ such that

$$(\mathrm{F}) \qquad (b'_1, b'_2, \cdots, b'_p) = \mu(b_1, b_2, \cdots, b_p) .$$

Also from the equalities (E_{i1}) we have numbers μ_i such that

$$(\mathrm{F}_i) \qquad (a'_i, b'_1) = \mu_i(a_i, b_1) .$$

But combining (F_i) with (F) and remembering that $b_1 \neq 0$ we have $\mu = b'_1/b_1 = \mu_i$. This shows $(a'_i, b'_j) = \mu(a_i, b_j)$ which would mean that $\mathscr{R}$ and $\mathscr{R}'$ are parallel. This proves $\mathscr{F}$ is SWC.

(iii) However, $\mathscr{F}$ is not SC. This can be proved as follows: Take any simple vector R which is linearly dependent on R_1, R_2, R_3 say $R = a_1R_1 + a_2R_2 + a_3R_3$ with $a_i \neq 0, i = 1$ to 3. Then we have $\mathscr{F}(R) = |a_1| + |a_2| + |a_3| = \mathscr{F}(a_1R_1) + \mathscr{F}(a_2R_2) + \mathscr{F}(a_3R_3)$, which violates strict inequality even on G_r^n. Consequently it is impossible to extend $\mathscr{F}$ to a strictly convex function on V_r^n. We note here that in the example of § 4 all vectors $a_1R_1 + a_2R_2 + (-a_1a_2/a_1 + a_2)R_3$ are simple. This completes the proof of Lemma B.

7. THEOREM. *The strict area minimizing property does not imply strict convexity.*

Proof. By Lemma A we have the $SFMA$ implied by SWC and C. But by Lemma B, SWC and C do not imply SC. Hence, the $SFMA$ does not imply SC. Briefly $SFMA \leqq SWC + C < SC$.

Suggestions for this paper from Professor H. Busemann are gratefully acknowledged.

REFERENCES

1. N. Bourbaki, *Algèbre Multilinéaire*, Elements de Math, I, livre II, Chap. 3, Paris, 1958.
2. H. Busemann, and E. G. Straus, *Area and normality*, Pacific J. Math. **10** (1960), 35–72.
3. H. Busemann, and G. C. Shephard, *Convexity on nonconvex sets*, Proc. Coll. Convexity, Copenhagen, 1965, (1967), 20–33.

Received October 16, 1968. This work was partially supported by the National Science Foundation under Grant No. GP-7472.

UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES