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HOLOMORPHIC QUADRATIC DIFFERENTIALS ON SURFACES IN E^3

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Let R be a Riemann surface defined upon an oriented surface S smoothly immersed in E^3. This paper studies holomorphic quadratic differentials on R which are related to the geometry on S, especially those of the form

$$\Omega_{\hat{A}} = \{(A - C) - 2iB\}dz^2$$

where $\hat{A} = Adx^2 + 2Bdxdy + Cdy^2$ is a smooth linear combination $\hat{A} = \hat{f}I + \hat{g}II$ of the fundamental forms on S, and $z = x + iy$ is any conformal parameter on R. Most results deal with the case in which $R = R_A$ is determined on S by some smooth positive definite linear combination $A = fI + gII$ on S. It is shown, for example, that S is isothermal with respect to A if and only if R_A supports a holomorphic $\Omega_{\hat{A}} \not\equiv 0$ in some neighborhood of any nonumbilic point. By way of contrast, another result states that a holomorphic $\Omega_{\hat{A}} \not\equiv 0$ is automatically available in the neighborhood of any nonumbilic point p, unless R coincides at p with some R_A. The paper closes with a study of surfaces which support an R_A on which both $\Omega_I \not\equiv 0$ and $\Omega_{II} \not\equiv 0$ are holomorphic.

Suppose that S is an oriented surface which is smoothly immersed in E^3, and that R is a Riemann surface defined upon the underlying 2-manifold of S so that each conformal parameter $z = x + iy$ on R yields a smooth, properly oriented coordinate pair x, y on S. As an example, $R = R_\theta$ might be determined on S by some sufficiently smooth positive definite quadratic form θ (see [1], § 4).

Ordinarily, there is no reason to expect any special relationship between the holomorphic quadratic differentials on R and the differential geometry on S. In particular, if one takes some quadratic form $\hat{\theta}$ of geometric interest on S, and associates with $\hat{\theta}$ the quadratic differential $\Omega_{\hat{\theta}} = \varphi_{\hat{\theta}} dz^2$ on R where $\hat{\theta} = Adx^2 + 2Bdxdy + Cdy^2$ and $\varphi_{\hat{\theta}} = (A - C) - 2iB$, then $\Omega_{\hat{\theta}}$ will not usually be holomorphic, that is, $\varphi_{\hat{\theta}}$ will not in general be analytic as a function of the conformal parameter $z = x + iy$ on R. There is always, of course, the trivial situation in which $\hat{\theta}$ is proportional to θ on S, so that $\Omega_{\hat{\theta}} \equiv 0$ is automatically holomorphic on R_θ. Yet in a striking number of cases, surfaces of particular interest to differential geometers have been shown to support a nontrivial holomorphic quadratic differential $\Omega_{\hat{\theta}} \not\equiv 0$ on some specific R. It seems appropriate to note a few such examples, as they provided the major motivation for the study undertaken in this paper.

First, Ω_{II} is holomorphic on R_I if mean curvature H on S is constant, with I and II the fundamental forms on S (see Chapter 4 of [2], or [3]). Here $\Omega_{II} \not\equiv 0$ so long as S is not totally umbilic. Analogously, Ω_I is holomorphic on R_{II} if Gauss curvature K is some positive constant, with S oriented so that H is positive (see [3]). Here $\Omega_I \not\equiv 0$ so long as S is not a portion of a sphere. If K is some negative constant on S, then $\Omega_{H'II}$ is holomorphic on $R_{II'}$ with $H' = \sqrt{H^2 - K} \neq 0$, and $H'II' = HII - KI$ (see [4] and [5]). Here $\Omega_{H'II} \equiv 0$ cannot occur. Finally, the immersion $X: S \to E^3$ is harmonic for some fixed R on S (meaning that $\Delta X \equiv 0$ holds on R) only if Ω_I is holomorphic on R. Here $\Omega_I \not\equiv 0$ unless S is a minimal surface (see [6]).

In the examples just described, θ and $\hat{\theta}$ were invariably smooth linear combinations of the fundamental forms on S. To distinguish such quadratic forms from more general θ and $\hat{\theta}$ on S, we shall reserve the symbols Λ and $\hat{\Lambda}$ to denote linear combinations $fI+gII$ and $\hat{f}I+\hat{g}II$ respectively, with $f, g, \hat{f}$ and $\hat{g}$ smooth, real valued functions. We further specify that Λ *must be positive definite*. No such restriction is placed upon $\hat{\Lambda}$.

In this paper, we attempt to describe the most general situations in which some $\Omega_{\hat{\Lambda}}$ on an R_Λ is holomorphic, and to develop the basic facts implied whenever such an $\Omega_{\hat{\Lambda}}$ and R_Λ are available on an S in E^3. The results should serve to encourage and facilitate the use of complex analysis in the solution of problems in surface theory. For, in each of the cases cited above, properties of the holomorphic $\Omega_{\hat{\Lambda}}$ involved have proved useful in handling questions (especially questions in-the-large) about S (see [2], [3], [6] or [8]).

Many of the technical lemmas below merely carry out in general arguments which had been separately justified for the various cases alluded to above. Other results help to clarify the special nature of Riemann surfaces of the form R_Λ on an S. In particular, we show that given an R on S, there is always a holomorphic $\Omega_{\hat{\Lambda}} \not\equiv 0$ available on R in some neighborhood of any nonumbilic point p on S, unless R coincides with some R_Λ at p, that is, unless R induces the same angle measurement at p as Λ does, for some Λ. By contrast, we show that S is isothermal with respect to Λ if and only if R_Λ supports a holomorphic quadratic differential $\Omega_{\hat{\Lambda}} \not\equiv 0$ in the neighborhood of any nonumbilic point. The paper closes with a brief discussion of those S in E^3 which support an R_Λ on which both $\Omega_I \not\equiv 0$ and $\Omega_{II} \not\equiv 0$ are holomorphic.

2. The results in this section pertain to structures R_Λ on an S in E^3. Quadratic differentials of the form $\Omega_{\hat{\Lambda}}$ will be mentioned, but they need not be holomorphic.

Given a positive definite quadratic form θ on S, coordinates x, y on S are called *θ-isothermal* provided that $\theta = \lambda(x, y)(dx^2 + dy^2)$ over the domain D of x, y on S. If $\Lambda = fI + gII$ is specified on S, one can always test given coordinates x, y to see if they are Λ-isothermal. In fact, x, y are Λ-isothermal if and only if

$$fE + gL = fG + gN > 0 \tag{1}$$

and

$$fF + gM = 0\ , \tag{2}$$

where $I = Edx^2 + 2Fdxdy + Gdy^2$ and $II = Ldx^2 + 2Mdxdy + Ndy^2$ over D. But if no Λ is specified, and x, y are given, one would like to have necessary and/or sufficient conditions for the existence of a Λ over D for which x, y are Λ-isothermal. Our first result establishes a necessary condition.

LEMMA 1. *If x, y are Λ-isothermal then*

$$M(E - G) = F(L - N) \tag{3}$$

holds throughout their domain D of definition.

Proof of Lemma 1. If x, y are Λ-isothermal for some $\Lambda = fI + gII$ over D, then at every point of D, f and g solve the homogeneous linear equations (1) and (2). Since Λ is positive definite f and g never vanish simultaneously. Thus the determinant of the coefficient matrix of the equations (1) and (2) must vanish, yielding (3).

It is a slightly more complicated matter to determine the circumstances under which (3) is sufficient for the existence of a

$$\Lambda = \lambda(x, y)(dx^2 + dy^2)$$

over D. Certainly (3) by itself is insufficient at an umbilic point, where the stronger conditions $E = G$ and $F = 0$ must be imposed. On the other hand, a more subtle obstacle is the need to make the choice of Λ smooth over D. Lemma 2 will show that under certain circumstances, Λ can be found provided that D avoids umbilics. To distinguish the nonumbilic portions of S, R, R_Λ, etc., we shall use the symbols S^0, R^0, R^0_Λ, etc., respectively. We further specify that the choice of signs in (4), (5) and similar formulas below be made so as to yield a positive definite (rather than a negative definite) quadratic form.

LEMMA 2. *Suppose that the domain D of coordinates x, y on S lies within S^0. If* (3) *holds with $|F| + |M| \neq 0$ throughout D, then*

x, y are Λ-isothermal for

$$\Lambda = \pm(MI - FII) . \tag{4}$$

If, on the other hand, x, y are lines-of-curvature coordinates on $D \subset S^0$, then x, y are Λ-isothermal for

$$\Lambda = \pm((L - N)I - (E - G)II) . \tag{5}$$

REMARK 1. Any coordinates x, y on S determine a unique Riemann surface R over their domain D of definition, the one upon which $z = x + iy$ is a conformal parameter. Thus the choices of Λ in Lemma 2 are uniquely determined by x, y up to multiplication by a smooth, positive function on D. In particular, lines-of-curvature coordinates x, y on S^0 uniquely determine an R_Λ over their domain of definition.

REMARK 2. It is convenient to speak of x, y as Λ-isothermal at p on S provided that $\Lambda = \lambda(dx^2 + dy^2)$ at p. Thus, for example, R_Λ coincides at p with the Riemann surface R on D determined by x, y (meaning that R_Λ and R induce the same angle measurement at p) if and only if x, y are Λ-isothermal at p. It is easily checked that x,y are Λ-isothermal at a nonumbilic p if and only if (3) holds there. They are Λ-isothermal at an umbilic p if and only if $E=G$ and $F=0$ for x, y at p.

Proof of Lemma 2. Since $D \subset S^0$, I is nowhere proportional to II on D. If (3) holds with $|F| + |M| \neq 0$ throughout D, then $MI+FII$ must have the form $\lambda(x, y)(dx^2 + dy^2)$ for an easily computed $\lambda(x, y)$ which never vanishes on D. If, on the other hand, x, y are lines-of-curvature coordinates, so that $F \equiv M \equiv 0$, then $(L - N)I - (E - G)II$ must have the form $\lambda(x, y)(dx^2 + dy^2)$ for an easily computed $\lambda(x, y)$ which never vanishes on D.

It is a natural extension of classical terminology to call S θ-*isothermal* for some smooth, positive definite quadratic form θ on S^0 provided that in some neighborhood of any $p \in S^0$ there exist lines-of-curvature coordinates x, y which are θ-isothermal. By way of example, surfaces of revolution and surfaces of constant mean curvature are I-isothermal, while surfaces of positive constant Gauss curvature are II-isothermal (see [3]).

Lemma 2 indicates that the portion of any S within the domain D of lines-of-curvature coordinates x, y on S^0 must be Λ-isothermal for the Λ given by (5). It is not clear that this Λ may be smoothly extended to all of S^0 so as to make S Λ-isothermal, for if x, y are lines-of-curvature coordinates, cx, y are also lines-of-curvature coordinates for any constant $c > 0$. Yet if $c \neq 1$, the Λ associated by (5) with x, y will not even be proportional to the Λ associated by (5) with cx, y. To distinguish a situation in which this difficulty can be overcome, we define the notion of a "coherent" covering of S^0.

A collection of coordinate pairs x, y whose domains cover a region $\mathscr{D} \subset S^0$ are said to cover $\mathscr{D}$ *coherently* (and yield a coherent covering of $\mathscr{D}$) provided that wherever the domains of covering pairs x, y and u, v intersect, the coefficients of I and II induced by x, y are identical (as point functions) with those induced by u, v. Thus Lemma 2 may be appropriately restated to yield a Λ over $\mathscr{D}$ for which all pairs x, y of a coherent covering of $\mathscr{D}$ are Λ-isothermal, provided that either (3) holds for all pairs of the covering with $|F| + |M|$ never zero, or that $F \equiv M \equiv 0$ holds for all covering pairs. The latter case has the following interpretation.

COROLLARY TO LEMMA 2. *If S^0 can be coherently covered by lines-of-curvature coordinates, then there is a Λ on S^0 for which S is Λ-isothermal.*

The next Lemma gives a convenient formula for directions of principal curvature on an R_Λ^0. Hopf noted in Chapter 4 of [2] that the usual equation (7) stated below has the form

$$\operatorname{Im} \Omega_{II} = 0$$

on R_I^0. We noted in [3] that (7) has the form

$$\operatorname{Im} \Omega_I = 0$$

on R_{II}^0. The following result indicates that these neat reformulations resulted solely from the fact that I is nowhere proportional to II on S^0.

LEMMA 3. *At any point of S where $\hat{\Lambda}$ is not proportional to Λ,*

$$\operatorname{Im} \Omega_{\hat{\Lambda}} = 0 \tag{6}$$

is the equation for directions of principal curvature on R_Λ.

Proof of Lemma 3. The assumption that $\hat{\Lambda}$ is not proportional to Λ guarantees that we are working at a point p on R_Λ^0. In terms of arbitrary coordinates x, y on S^0 near p, the equation for directions of principal curvature (see [9], p. 80) is

$$(EM - FL)dx^2 + (FN - GM)dy^2 + (EN - GL)dxdy = 0 \,. \tag{7}$$

If x, y are Λ-isothermal at p, then (3) holds, so that (7) becomes

$$(FN - GM)(dy^2 - dx^2) + (EN - GL)dxdy = 0 \,. \tag{8}$$

But since $\hat{\Lambda} = \hat{f}I + \hat{g}II$, (6) may be rewritten to read

$$(9) \quad (\hat{f}F + \hat{g}M)(dy^2 - dx^2) + \{\hat{f}(E - G) + \hat{g}(L - N)\}dxdy = 0 \,.$$

It remains to show that (8) and (9) are equivalent.

Suppose first that $FN = GM$ at p. Because (3) holds with $G \neq 0$ and $I \not\propto II$, we conclude that $EN - GL \neq 0$. But then (8) reduces to $dxdy = 0$, making $F = M = 0$ at p. On the other hand, $\varphi_{\hat{\Lambda}} \neq 0$ since $\hat{\Lambda} \not\propto \Lambda$. Thus $\operatorname{Re} \varphi_{\hat{\Lambda}} = \hat{f}F + \hat{g}M = 0$ implies that

$$\operatorname{Im} \varphi_{\hat{\Lambda}} = \{\hat{f}(E - G) + \hat{g}(L - N)\} \neq 0 \,.$$

But then (9), like (8), reduces to $dxdy = 0$.

Suppose next that $FN \neq GM$ at p. Using (3) and $I \not\propto II$, we conclude that $\hat{f}F + \hat{g}M \neq 0$. But then both sides of (8) may be multiplied by

$$\frac{\hat{f}F + \hat{g}M}{FN - GM} \neq 0$$

to obtain (9), a clearly reversible process.

REMARK 3. Given a quadratic differential $\Omega = \varphi dz^2$ on any Riemann surface R, the integral curves of the directions $\varphi dz^2 > 0$ and $\varphi dz^2 < 0$ (well defined wherever $\varphi \neq 0$) are known as the trajectories and orthogonal trajectories of Ω respectively. Lemma 3 thus states that *wherever $\hat{\Lambda}$ is not proportional to Λ on S, the trajectories and orthogonal trajectories of $\Omega_{\hat{\Lambda}}$ on R_Λ are the lines of curvature on S^0.* Thus the following result is not at all surprising.

COROLLARY TO LEMMA 3. *If $\hat{\Lambda}$ is nowhere proportional to Λ in some deleted neighborhood of p on S, then the index of p in the net of lines of curvature is given by*

$$(10) \qquad i(p) = \frac{-1}{4\pi} \Delta_C \quad \arg \quad \varphi_{\hat{\Lambda}}$$

where the change in argument is taken going once in the positive sense about any sufficiently small conformal parameter circle C on R_Λ centered at p.

Proof of Corollary to Lemma 3. If z is a conformal parameter on R_Λ near p, and if $z = z_0$ at p, choose for C any circle $|z - z_0| = \varepsilon > 0$ so small that $\hat{\Lambda}$ is nowhere proportional to Λ on or inside C, except perhaps at p itself. Then (see Chapter 3 of [2])

$$i(p) = \frac{1}{2\pi} \Delta_C \quad \arg \quad dz$$

where $dz = dx + idy$ solves (6), and where C is traversed once in the positive sense. But (6) gives

$$\arg dz = \frac{m\pi}{2} - \frac{1}{2} \arg \varphi_{\hat{A}}$$

with m an integer, so that (10) follows easily.

In closing this section, we note that there are Riemann surfaces R available on any S in E^3 which are not of the form R_A for any A. In fact, one may specify any $p \in S$, and find an R_θ on S which is not of the form R_A for any A at any point in some neighborhood of p. This can be done since the positive definite forms $fI + gII$ at p constitute at most a 2-dimensional subset of the 3-dimensional space of all positive definite quadratic forms at p. But any θ not of the form A at p can be smoothly extended to a positive definite quadratic form θ on S, in which case, θ is not of the form A for any A at any point in some neighborhood of p. Of course, one cannot in general find an R on S which at no point of S is of the form R_A. If, for example, S is compact with genus $h \neq 1$, any R must coincide somewhere on S with any given R_A. Otherwise, the identity map $i: R \to R_A$ would preserve a unique pair of orthogonal directions at every point of S, yielding a nonvanishing tangential direction field on S, a contradiction.

3. In this section, we begin the study of holomorphic quadratic differentials $\Omega_{\hat{A}}$. As an immediate consequence of Lemma 3 and its corollary, we have the following result.

LEMMA 4. *If $\Omega_{\hat{A}} \not\equiv 0$ on R_A is holomorphic, then the zeros of $\Omega_{\hat{A}}$ coincide with the umbilics on S, and any umbilic p has index*

$$i(p) = \frac{-n}{2} \tag{11}$$

where n is the order of the zero of $\Omega_{\hat{A}}$ at p.

Proof of Lemma 4. At any umbilic p, $A \propto \hat{A}$ yields $\Omega_{\hat{A}}(p) = 0$. Since the zeros of any nontrivial holomorphic quadratic differential are isolated, any point p on S has a deleted neighborhood within S^0. Thus $i(p)$ may be computed at any $p \in S$ using (10). But, because $\varphi_{\hat{A}}$ is analytic, the order n of the zero (if any) of $\Omega_{\hat{A}}$ at p is given by

$$n = \frac{1}{2\pi} \Delta \arg \varphi_{\hat{A}}, \tag{12}$$

which yields (11). If p is a zero of $\Omega_{\hat{A}}$, $i(p) \neq 0$ forces p to be an umbilic on S^0. We have also established the following.

COROLLARY TO LEMMA 4. *If $\Omega_{\hat{A}} \not\equiv 0$ on R_A is holomorphic, then the umbilics on S are isolated, irremovable, and have negative index.*

Suppose now, more generally, that $\Omega = \varphi dz^2$ is a meromorphic quadratic differential on some R defined on S, that is, suppose each φ is meromorphic in its defining conformal parameter z on R. Then the trajectories and orthogonal trajectories of Ω are well defined wherever $\Omega \neq 0$ and $\Omega \neq \infty$. Suppose that wherever the trajectories and orthogonal trajectories of Ω are defined, they are lines of curvature *on* S^0. This specifically includes the *assumption* that points where $\Omega \neq 0$ and $\Omega \neq \infty$ are *in* S^0. Then

$$\operatorname{Im} \Omega > 0$$

remains the equation for directions of principal curvature on R wherever $\Omega \neq 0$ and $\Omega \neq \infty$. But the zeros and poles of a meromorphic Ω must be isolated. Thus (10) applies to compute $i(p)$ for any p on S, with φ in place of $\varphi_{\hat{A}}$. Moreover, (12) may be stated for φ, with n the order of the zero or minus the order of the pole of Ω at p, $n = 0$ meaning that neither a zero nor pole occurs. The following result is thus easy to check.

LEMMA 5. *Suppose $\Omega \not\equiv 0$ on R is a meromorphic quadratic differential whose trajectories and orthogonal trajectories (wherever they are defined) are lines of curvature on S^0. Then the umbilics on S coincide with the zeros and poles of Ω, and any umbilic p has index $i(p)$ given by* (11) *where n is the order of the zero or minus the order of the pole of Ω at p.*

COROLLARY TO LEMMA 5. *Under the hypotheses of Lemma* 5, *the umbilics on S are isolated and irremoveable.*

A recent result of Titus (see [11]) indicates that the index $i(p)$ of an isolated umbilic on a surface smoothly immersed in E^3 must satisfy $i(p) \leqq 1$. (For old proofs assuming real analyticity of S near p, see references 2, 3 and 5 listed in [3]). Thus the next statement can be made.

COROLLARY′ TO LEMMA 5. *Under the hypotheses of Lemma* 5, *any pole of Ω has order less than* 3.

The next result indicates how little extra information was gained in going from Lemma 4 to Lemma 5. For, given the hypotheses of Lemma 5, and deleting from S the poles of Ω, one is in the situation covered by Lemma 4, except, perhaps, at the zeros of Ω.

LEMMA 6. *Suppose that $\Omega \not\equiv 0$ is a holomorphic quadratic differential on R whose trajectories and orthogonal trajectories are lines of curvature on S^0. Then Λ and $\hat{\Lambda}$ exist on S^0 such that $\Omega = \Omega_{\hat{\Lambda}}$ on R^0, and $R^0 = R^0_\Lambda$.*

Proof of Lemma 6. By Lemma 5, Ω never vanishes on R^0. But near any point which is not a zero of Ω, there is a distinguished conformal parameter $z = x + iy$ on R (so that x, y are called distinguished coordinates on S for Ω on R) in terms of which $\Omega = dz^2$ (see [1], p. 103). Such distinguished parameters for Ω on R are uniquely determined over their domains of definition up to addition of a complex constant, or multiplication by -1. Moreover $x \equiv$ constant and $y \equiv$ constant are the equations for the trajectories and orthogonal trajectories of Ω over the domain of any distinguished $z = x + iy$ on R. Thus, given the hypotheses of Lemma 6, it follows that the distinguished coordinates x, y on S for Ω on R constitute a coherent covering of S^0 by lines-of-curvature coordinates. Now (5) provides a Λ on S^0 such that $R^0 = R^0_\Lambda$, and the choice

$$\hat{\Lambda} = \left\{\frac{Gk_2 + Ek_1}{EG(k_2 - k_1)}\right\} I - \left\{\frac{E + G}{EG(k_2 - k_1)}\right\} II$$

for all distinguished x, y yields a well defined $\hat{\Lambda}$ over S^0 in terms of which $\Omega_{\hat{\Lambda}} = \Omega$ on R^0.

To complement Lemma 6, the next result indicates that any holomorphic quadratic differential $\Omega \neq 0$ over an R on S is of the form $\Omega_{\hat{\Lambda}}$ on R^0, unless R at some point p on R^0 coincides with an R_Λ for some Λ.

LEMMA 7. *Suppose R is not of the form R_Λ for any Λ at any point of R^0. Suppose that $\Omega \neq 0$ is holomorphic on R^0. Then there exists a uniquely determined $\hat{\Lambda}$ on S^0 such that $\Omega_{\hat{\Lambda}} = \Omega$ on R^0.*

Proof. Since Ω never vanishes, the distinguished coordinates x, y on S for Ω on R^0 provide a coherent covering of S^0. Thus, using just these distinguished x, y,

$$\hat{\Lambda} = \frac{MI}{(E - G)M - (L - N)F} - \frac{FII}{(E - G)M - (L - N)F}$$

is well defined over S^0 where, by Remark 2, (3) is never satisfied. It is trivial to check that $\Omega_{\hat{\Lambda}} = \Omega$ on R^0. To show that $\hat{\Lambda}$ is uniquely determined at every p on S^0, suppose that $\tilde{\Lambda} = \tilde{f}I + \tilde{g}II$ also yields

$\Omega_{\hat{\Lambda}} = \Omega$ at p. Then

$$\Omega_{(\hat{\Lambda}-\tilde{\Lambda})}(p) = 0$$

so that $\hat{\Lambda} - \tilde{\Lambda} = (\hat{f} - \tilde{f})I + (\hat{g} - \tilde{g})II$ must be of the form $\lambda(dx^2 + dy^2)$ at p. Thus, unless $\lambda = 0$, the positive definite form

$$\Lambda = \pm(\hat{\Lambda} - \tilde{\Lambda})$$

gives rise to an $R_\Lambda = R$ at p, a contradiction.

If Lemma 7 is applied in some neighborhood of p on S^0 where R is nowhere of the form R_Λ for any Λ, then taking any z on R near p and setting $\Omega = dz^2$, we obtain the following.

COROLLARY TO LEMMA 7. *In some neighborhood of any p on S^0 at which R is not of the form R_Λ for any Λ, there exists a holomorphic quadratic differential $\Omega_{\hat{\Lambda}} \not\equiv 0$.*

If $\Omega_{\hat{\Lambda}} \not\equiv 0$ is holomorphic on an arbitrary R on S rather than on an R_Λ, then one loses the identification of umbilics on S with the zeros of $\Omega_{\hat{\Lambda}}$ provided by Lemma 4. Nonetheless, the following can be said.

LEMMA 8. *If $\Omega_{\hat{\Lambda}} \not\equiv 0$ on R is holomorphic with $\Omega_{\hat{\Lambda}}(p) = 0$, then either p is an umbilic, or else $R = R_\Lambda$ for some Λ at p, or else $\hat{f} = \hat{g} = 0$ at p for $\hat{\Lambda}$.*

Proof of Lemma 8. For any $z = x + iy$ on R near p,

$$\begin{aligned} \hat{f}(E - G) + \hat{g}(L - N) &= 0 \\ \hat{f}F + \hat{g}M &= 0 \end{aligned} \tag{13}$$

holds at p since $\Omega_{\hat{\Lambda}}$ has a zero there. If $|\hat{f}| + |\hat{g}| \neq 0$ at p, the determinant of the coefficient matrix for (13) must vanish at p yielding (3). Thus, unless p is an umbilic, Remark 2 indicates that $R = R_\Lambda$ at p for some Λ.

In contrast with the situation described in Lemma 7 and its corollary, our next results indicate that severe restrictions are placed upon S in claiming (even locally) the existence of a holomorphic $\Omega_{\hat{\Lambda}} \not\equiv 0$ on an R_Λ. Moreover, even if S does support such a holomorphic $\Omega_{\hat{\Lambda}} \not\equiv 0$ on some R_Λ, most holomorphic $\Omega \not\equiv 0$ on R_Λ will not be of the form $\Omega_{\hat{\Lambda}}$.

THEOREM 1. *If $\Omega_{\hat{\Lambda}} \not\equiv 0$ on R_Λ is holomorphic, then S is Λ-isothermal.*

Proof of Theorem 1. At any $p \in S^0$, $\Omega_{\hat{\Lambda}}(p) \neq 0$. Thus, using distinguished coordinates x, y near p for $\Omega_{\hat{\Lambda}}$ on R_Λ, $\hat{\Lambda}$ must be of the form

$$\hat{\Lambda} = Adx^2 + (A-1)dy^2 ,$$

while

$$\Lambda = \lambda(dx^2 + dy^2) .$$

But $\Lambda = fI + gII$ and $\hat{\Lambda} = \hat{f}I + \hat{g}II$ yield

$$fF + gM = 0 , \qquad \hat{f}F + \hat{g}M = 0 \tag{14}$$

for x, y. Moreover, $\Omega_{\hat{\Lambda}} = dz^2$ near p shows that $\hat{\Lambda}$ is never proportional to Λ in some neighborhood of p where, therefore, $f\hat{g} - g\hat{f} \neq 0$. It follows that (14) has there only the trivial solution $F \equiv M \equiv 0$. Thus near any p on S^0, there exist Λ-isothermal lines-of-curvature coordinates x, y. Combining Lemmas 5 and 6 with Theorem 1, we can state the following.

COROLLARY TO THEOREM 1. *If $\Omega \not\equiv 0$ is a meromorphic quadratic differential on R whose trajectories and orthogonal trajectories are lines-of-curvature on S^0, then there exists a Λ on S^0 for which S is Λ-isothermal.*

THEOREM 2. *If S^0 can be coherently covered by Λ-isothermal lines-of-curvature coordinates, then there exists a $\hat{\Lambda}$ on S^0 for which $\Omega_{\hat{\Lambda}} \neq 0$ on R^0_Λ is holomorphic.*

Proof of Theorem 2. Define a holomorphic quadratic differential $\Omega \neq 0$ on R^0_Λ as follows. Use a coherent covering of S^0 by isothermal lines-of-curvature coordinates. If x, y and u, v are covering pairs whose domains intersect, then over that intersection D, $w = u + iv$ is an analytic function of $z = x + iy$ since w and z are both conformal parameters on R_Λ. But the correspondence of curves $x \equiv$ constant and $y \equiv$ constant to curves $u \equiv$ constant and $v \equiv$ constant implies that $w = cz + d$ with c real or pure imaginary. That $c = \pm 1$ follows from the fact that $k_1 \neq k_2$ on S^0, while $E, F = 0, G, L = k_1E, M = 0$ and $N = k_2G$ are the same for x, y over D as for u, v. Thus $dz^2 = dw^2$ on D, and the consistent choice of $\Omega = dz^2$ for covering coordinates x, y yields a holomorphic Ω which never vanishes on S^0 and whose trajectories are lines of curvature on S^0. Lemma 6 now applies with $R = R^0_\Lambda$, giving the required $\hat{\Lambda}$ on S^0.

Of course, if S is Λ-isothermal, then in some neighborhood of any p on S^0 there is a $\hat{\Lambda}$ for which $\Omega_{\hat{\Lambda}} \neq 0$ on R_Λ is holomorphic. This together with Theorem 1 yields the following.

CHARACTERIZATION. *Given Λ on S^0, S is Λ-isothermal if and* *only if in some neighborhood of any p on S^0 there is a $\hat{\Lambda}$ for which $\Omega_{\hat{\Lambda}} \neq 0$ on R_Λ is holomorphic.*

4. In this section we make a detailed study of functions on S^0 associated with a holomorphic $\Omega_{\hat{\Lambda}} \not\equiv 0$ on an R_Λ. Given such a $\Omega_{\hat{\Lambda}}$ on a specified R_Λ, Lemma 4 applies, and one easily checks that the distinguished coordinates for $\Omega_{\hat{\Lambda}}$ on R_Λ^0 provide a coherent covering of S^0 by Λ-isothermal lines-of-curvature coordinates. This covering determines functions E and G on S^0 such that, given distinguished coordinates x, y on S^0 for $\Omega_{\hat{\Lambda}}$ on R_Λ^0,

$$\begin{aligned} I &= E dx^2 + G dy^2 \\ II &= k_1 E dx^2 + k_2 G dy^2 \,. \end{aligned} \tag{15}$$

Note that knowledge of E, G, k_1 and k_2 on S^0 will not by itself determine I and II over S^0. For this purpose, one must also have some way of recognizing which coordinates x, y on S^0 are distinguished for $\Omega_{\hat{\Lambda}}$ on R_Λ^0. The next lemma allows us to work with normalized versions of Λ and $\hat{\Lambda}$ throughout S^0.

LEMMA 8. *If $\Omega_{\hat{\Lambda}} \not\equiv 0$ on R_Λ is holomorphic, then there exist quadratic forms Σ and $\hat{\Sigma}$ on S^0 which are smooth linear combinations of I and II, and which have the form $\Sigma = dx^2 + dy^2$ and $\hat{\Sigma} = (1/2)(dx^2 - dy^2)$ in terms of distinguished coordinates x, y for $\Omega_{\hat{\Lambda}}$ on R_Λ, so that $R_\Lambda = R_\Sigma$ and $\Omega_{\hat{\Lambda}} = \Omega_{\hat{\Sigma}}$.*

Proof of Lemma 8. The choices indicated locally for Σ and $\hat{\Sigma}$ in terms of distinguished coordinates for $\Omega_{\hat{\Lambda}}$ on R_Λ^0 yield well defined quadratic forms Σ and $\hat{\Sigma}$ over all of S^0. For, as noted in the proof of Theorem 2, where the domains of distinguished parameters $w = u + iv$ and $z = x + iy$ intersect, $w = \pm z + d$, yielding $dx^2 = du^2$ and $dy^2 = dv^2$. We need only check therefore over the domain D of any distinguished coordinate pair x, y that $R_\Lambda = R_\Sigma$, that $\Omega_{\hat{\Lambda}} = \Omega_{\hat{\Sigma}}$ and that both Σ and $\hat{\Sigma}$ are smooth linear combinations of I and II. But these are indicated by noting that $\Omega_{\hat{\Sigma}} = dz^2$ over D, that $\Lambda = fI + gII = \lambda(dx^2 + dy^2)$ over D, and that $\hat{\Lambda} = A dx^2 + (A - 1) dy^2$ over D, yielding

$$\Sigma = \frac{f}{\lambda} I + \frac{g}{\lambda} II$$

and

$$\hat{\Sigma} = \hat{\Lambda} + \frac{1 - 2A}{2} \Sigma$$

over D.

We conclude from Lemma 8 that there is no loss of generality in working on S^0 from the outset with the *normalized* forms

$$\Lambda = fI + gII = \Sigma \tag{16}$$

and

$$\hat{\Lambda} = \hat{f}I + \hat{g}II = \hat{\Sigma} \; . \tag{17}$$

With Λ and $\hat{\Lambda}$ so chosen, the smooth functions $f, g, \hat{f}$ and $\hat{g}$ are well defined on S^0, The functions E, G, k_1 and k_2 associated with the coherent covering of S^0 by distinguished coordinates for $\Omega_{\hat{\Lambda}}$ on R^0_Λ are, of course, related to the functions $f, g, \hat{f}$ and $\hat{g}$. The precise relationship is given in the following result.

LEMMA 9. *The functions $E, G, k_1, k_2, f, g, \hat{f}$ and $\hat{g}$ on S^0 satisfy*

$$f + k_1 g - \frac{1}{E} = 0 \; , \tag{18}$$

$$f + k_2 g - \frac{1}{G} = 0 \; , \tag{19}$$

$$\hat{f} + k_1 \hat{g} - \frac{1}{2E} = 0 \; , \tag{20}$$

$$\hat{f} + k_2 \hat{g} + \frac{1}{2G} = 0 \; . \tag{21}$$

Proof of Lemma 9. Substitute (15) in (16) and (17). Then equate the coefficients of dx^2 and dy^2 respectively on opposite sides of the two resulting equations.

In reading (18)-(21) it is well to keep in mind that

$$k_1 \neq k_2, f^2 + g^2 \neq 0, \hat{f}^2 + \hat{g}^2 \neq 0, \hat{f}g - f\hat{g} \neq 0 \; ,$$

and $E > 0, G > 0$. One may also use (19) and (21) to show that $2\hat{g} \pm g \neq 0$ since, otherwise, $f\hat{g} - \hat{f}g = 0$ would follow. As a direct consequence of (18)-(21) we have the following.

LEMMA 10. *Throughout S^0*

$$\begin{aligned} E &= \frac{2\hat{g} - g}{2(\hat{g}f - \hat{f}g)} \; , \qquad & G &= \frac{2\hat{g} + g}{2(\hat{g}f - g\hat{f})} \; , \\ k_1 &= \frac{2\hat{f} - f}{g - 2\hat{g}} \; , \qquad & k_2 &= \frac{-2\hat{f} - f}{\hat{g} + 2\hat{g}} \; , \end{aligned} \tag{22}$$

while

$$(23)\qquad \begin{aligned} f &= \frac{Gk_2 - Ek_1}{EG(k_2 - k_1)}\,, & g &= \frac{E - G}{EG(k_2 - k_1)}\,, \\ \hat{f} &= \frac{Gk_2 + Ek_2}{2EG(k_2 - k_1)}\,, & \hat{g} &= \frac{E + G}{2EG(k_1 - k_2)}\,. \end{aligned}$$

REMARK 4. Lemma 10 indicates that throughout S^0

$$K = k_1k_2 = \frac{f^2 - 4\hat{f}^2}{g^2 - 4\hat{g}^2}\,, \qquad H = \frac{k_1 + k_2}{2} = \frac{4\hat{f}\hat{g} - fg}{g^2 - 4\hat{g}^2}$$

and

$$H' = \left|\frac{k_1 - k_2}{2}\right| = 2\,\frac{g\hat{f} - f\hat{g}}{g^2 - 4\hat{g}^2}\,.$$

Thus in particular the expression $f^2 - 4\hat{f}^2/g^2 - 4\hat{g}^2$ for K is completely determined by I on S^0.

Various simple facts may be read from the equations (22) and (23). We list a few.

Fact 1. $f + k_1g, f + k_2g, \hat{f} + k_1\hat{g}$ and $\hat{f} - k_2\hat{g}$ are all positive.
Fact 2. If $k_1 = 0$, f and $\hat{f}$ are positive.
Fact 3. $2\hat{f} - f = 0$ if and only if $k_1 = 0$.
Fact 4. $2\hat{f} + f = 0$ if and only if $k_2 = 0$.
Fact 5. Sign $\hat{g}$ = Sign $(k_1 - k_2) \neq 0$.
Fact 6. If $\hat{f} = 0$, $K < 0$.
Fact 7. If $f = 0$, then $K > 0$.
Fact 8. If $H = 0$, then $f > 0$.
Fact 9. Sign $(2\hat{g} - g)$ = Sign $(2\hat{g} + g)$ = Sign $(f\hat{g} - \hat{f}g) \neq 0$.

Lemma 10 suggests the possibility of expressing any four of the functions $k_1, k_2, E, G, f, g, \hat{f}$ and $\hat{g}$ in terms of the remaining four. While this is not always possible, simple arithmetic will establish the following result.

LEMMA 11. *Given any* 4-*tuple from among the functions* $k_1, k_2, E, G, f, g, \hat{f}$ *and* $\hat{g}$ *on* S^0 *which is included in the chart printed at the top of the next page, equations* (18)-(21) *may be uniquely solved for the remaining* 4-*tuple* (*subject to parenthesized hypotheses*).

REMARK 5. By Fact 5, H, K and Sign $\hat{g}$ determine k_1 and k_2 over S^0. Thus any of the 4-tuples $K, H, \hat{g}, E$, or $HK\hat{g}G$ or $HK\hat{g}g$ or $HK\hat{g}\hat{f}$, or (if $H \neq 0$) H, K, g, f will determine all 8 of the functions involved in (18)-(21) over S^0.

Suppose now that D is a simply connected domain in the x, y-plane. If $E > 0, G > 0$, k_1 and $k_2 \neq k_1$ are any smooth functions on D, the fundamental theorem of surface theory (see [9], p. 124 and p. 113)

k_1k_2EG	$k_1Ef\hat{g}$ $(if\ k_1 \neq 0)$	$k_2Gf\hat{g}$ $(if\ k_2 \neq 0)$
k_1k_2Ef $(if\ k_1 \neq 0)$	$k_1Eg\hat{f}$ $(if\ k_1 \neq 0)$	$k_2Gg\hat{f}$ $(if\ k_2 \neq 0)$
k_1k_2Eg	$k_1Eg\hat{g}$	$k_2Gg\hat{g}$
$k_1k_2E\hat{f}$ $(if\ k_1 \neq 0)$	k_1Gfg $(if\ g \neq 0)$	$k_2fg\hat{f}$ $(if\ k_2 \neq 0)$
$k_1k_2E\hat{g}$	$k_1Gf\hat{f}$ $(if\ k_1 \neq 0)$	$k_2fg\hat{g}$
k_1k_2Gf $(if\ k_2 \neq 0)$	$k_1Gg\hat{g}$	$k_2f\hat{f}\hat{g}$ $(if\ k_2 \neq 0)$
k_1k_2Gg	$k_1G\hat{f}\hat{g}$	$k_2g\hat{f}\hat{g}$
$k_1k_2G\hat{f}$ $(if\ k_2 \neq 0)$	$k_1fg\hat{f}$ $(if\ k_1 \neq 0)$	$EGfg$ $(if\ E \neq G)$
$k_1k_2G\hat{g}$	$k_1fg\hat{g}$	$EGf\hat{g}$ $(if\ E \neq G)$
k_1k_2fg	$k_1f\hat{f}\hat{g}$ $(if\ k_1 \neq 0)$	$EGg\hat{f}$ $(if\ E \neq G)$
$k_1k_2f\hat{f}$ $(if\ K \neq 0)$	$k_1g\hat{f}\hat{g}$	$EG\hat{f}\hat{g}$
$k_1k_2f\hat{g}$ $(if\ H \neq 0)$	k_2EGf $(if\ k_2 \neq 0\ and\ E \neq G)$	$Efg\hat{f}$ $(if\ g \neq 0)$
$k_1k_2g\hat{f}$ $(if\ H \neq 0)$	k_2EGg $(if\ E \neq G)$	$Efg\hat{g}$ $(if\ g \neq 0)$
$k_1k_2g\hat{g}$	$k_2Eg\hat{f}$ $(if\ k_2 \neq 0)$	$Ef\hat{f}\hat{g}$ $(if\ 2\hat{f} \neq f)$
$k_1k_2\hat{f}\hat{g}$	$k_2EG\hat{g}$	$Eg\hat{f}\hat{g}$
k_1EGf $(if\ k_1 \neq k_2\ and\ E \neq G)$	k_2Efg $(if\ g \neq 0)$	$Gfg\hat{f}$ $(if\ g \neq 0)$
k_1EGg $(if\ E \neq G)$	$k_2Ef\hat{f}$ $(if\ k_2 \neq 0\ and\ Ef \neq 1)$	$Gfg\hat{g}$ $(if\ g \neq 0)$
$k_1Eg\hat{f}$ $(if\ k_1 \neq 0)$	$k_2Eg\hat{g}$	$Gf\hat{f}\hat{g}$ $(if\ 2\hat{f} \neq -f)$
$k_1EG\hat{g}$	$k_2E\hat{f}\hat{g}$	$Gg\hat{f}\hat{g}$
$k_1Ef\hat{f}$ $(if\ k_1 \neq 0)$	$k_2Gf\hat{f}$ $(if\ k_2 \neq 0)$	$fg\hat{f}\hat{g}$.

and the Fact in [7] state that D is smoothly immersible in E^3 with $I = Edx^2 + Gdy^2$ if and only if

$$k_1k_2 = \frac{-1}{2\sqrt{EG}}\left\{\left(\frac{G_x}{\sqrt{EG}}\right)_x + \left(\frac{E_y}{\sqrt{EG}}\right)_y\right\} \tag{24}$$

and

$$(k_1)_y = \frac{(k_2 - k_1)E_y}{2E}\,, \qquad (k_2)_x = \frac{(k_1 - k_2)G_x}{2G}\,. \tag{25}$$

Using Λ given by (5), and $\hat{\Lambda}$ from the proof of Lemma 6, (24) and (25) become necessary and sufficient conditions for the smooth immersion of D in E^3 with x, y distinguished coordinates for $\Omega_{\hat{\Lambda}} = dz^2$ which is holomorphic on R_Λ.

On the other hand, suppose we take any four smooth functions α, β, γ and δ on D, and assign them the role of a 4-tuple σ from the formal list $E, G, k_1, k_2, f, g, \hat{f}$ and $\hat{g}$, subject only to the restriction that given the values α, β, γ and δ in their σ-roles, (18)-(21) may be uniquely solved for any of the functions E, G, k_1 and k_2 not already in σ so as to yield over D an $E > 0, G > 0$ and a $k_1 \neq k_2$. Then using these values for E, G, k_1 ond k_2, (24) and (25) constitute necessary and sufficient conditions upon α, β, γ and δ for the smooth immersion of D in E^3 with α, β, γ and δ achieving their σ-roles, with

$$I = Edx^2 + Gdy^2, II = k_1 Edx^2 + k_2 Gdy^2, \Lambda = fI + gII$$
$$= dx^2 + dy^2, \hat{\Lambda} = \hat{f}I + \hat{g}II = \tfrac{1}{2}(dx^2 - dy^2) \ ,$$

and with $f, g, \hat{f}$ and $\hat{g}$ satisfying (23). To illustrate the proceedure, we state the following result.

THEOREM 3. *Suppose $f, g, \hat{f}$ and $\hat{g}$ are smooth functions on a simply connected domain D in the xy-plane such that*

$$\text{Sign}\,(2\hat{g} - g) = \text{Sign}\,(2\hat{g} + g) = \text{Sign}\,(f\hat{g} - \hat{f}g) \neq 0 \ . \tag{26}$$

Then (24) *and* (25) *with E, G, k_1 and k_2 given by* (22) *are necessary and sufficient conditions for the smooth immersion of D in E^3 with $\Lambda = fI + gII = dx^2 + dy^2$, $\hat{\Lambda} = \hat{f}I + \hat{g}II = \frac{1}{2}(dx^2 - dy^2)$, $I = Edx^2 + Gdy^2$ and $II = k_1 Edx^2 + k_2 Gdy^2$, making $\Omega_{\hat{\Lambda}} = dz^2$ on R_Λ over D.*

Proof of Theorem 3. The necessity is obvious. Sufficiency is established by noting that (26) insures values for E, G, k_1 and k_2 in (22) such that $E > 0$, $G > 0$, and $k_1 \neq k_2$. Simple arithmetic will show that any immersion of D in E^3 with I and II as specified will yield $fI + gII = dx^2 + dy^2$ and $\hat{f}I + \hat{g}II = \frac{1}{2}(dx^2 - dy^2)$. Finally, the smooth immersion of D in E^3 with I and II as given satisfying (24) and (25) is guaranteed by the Fact in [7].

5. Our basic method throughout § 4 was simple arithmetic. No use was made of the Gauss or Codazzi-Mainardi equations, except to say, of course, that they do describe the immersion of S in E^3. In practical terms however, these equations are the most likely tools for finding in any particular situation exactly which $\Omega_{\hat{\Lambda}} \not\equiv 0$ (if any) is holomorphic, and on which R_Λ (see Chapter 4 of [2], or [3] as examples).

For the various cases described in § 1, interest in S preceeded discovery of $\Omega_{\hat{\Lambda}}$ on R_Λ. But one can work in the other direction, and seek to discover by use of the Gauss and/or Codazzi-Mainardi equations which sort of S will support a particular $\Omega_{\hat{\Lambda}} \not\equiv 0$ which is holomorphic on a specified R_Λ. As a simple variation on this proceedure, we study in this section the nature of an S in E^3 which supports a Riemann surface R_Λ on which both $\Omega_I \not\equiv 0$ and $\Omega_{II} \not\equiv 0$ are holomorphic. We *assume henceforth that S is such a surface.*

LEMMA 12. *In the neighborhood of any point on S^0 there exist coordinates x, y on R_Λ in terms of which*

$$I = \frac{1 - ck_2}{k_1 - k_2}dx^2 + \frac{1 - ck_1}{k_1 - k_2}dy^2$$

and

$$II = \frac{k_1(1 - ck_2)}{k_1 - k_2} dx^2 + \frac{k_2(1 - ck_1)}{k_1 - k_2} dy^2$$

for some fixed constant $c \neq 0$.

Proof of Lemma 12. Use distinguished coordinates x, y for $\Omega_{II} \not\equiv 0$ on R_A to give E, G, k_1 and k_2 associated with them anywhere on S^0. In terms of such x, y,

$$Ek_1 - Gk_2 = 1 \tag{27}$$

while

$$\Omega_I = (E - G)dz^2 \not\equiv 0 ,$$

so that $(E - G)$ must be analytic in $z = x + iy$. It follows, since $(E - G)$ is also real valued, that

$$(E - G) = c \neq 0 \tag{28}$$

over the domain of x, y for some real constant c. Solution of (27) and (28) for E and G yields

$$E = \frac{1 - ck_2}{k_1 - k_2} , \qquad G = \frac{1 - ck_1}{k_1 - k_2} . \tag{29}$$

That c maintains a fixed value throughout S^0 follows from the fact that S^0 has one component since, by Lemma 4, umbilics on S are isolated. We note in passing that $\Omega_I = c\Omega_{II}$ on R_A, while

$$\text{Sign}\,(1 - ck_1) = \text{Sign}\,(1 - ck_2) = \text{Sign}\,(k_1 - k_2) \neq 0 \tag{30}$$

throughout S^0.

LEMMA 13. *S is a Weingarten surface satisfying*

$$(1 - ck_1)(1 - ck_2) = \mathscr{C} , \tag{31}$$

or (*equivalently*)

$$1 - 2cH + c^2K = \mathscr{C} \tag{31'}$$

for constants $c \neq 0$ *and* $\mathscr{C} > 0$.

Proof of Lemma 13. Using the coordinates x, y and the constant c of Lemma 12 anywhere on S^0, the Codazzi-Mainardi equations (25) with E and G given by (29) yield

$$\frac{(k_1)_y}{1 - ck_1} + \frac{(k_2)_y}{1 - ck_2} = 0$$

and

$$\frac{(k_1)_x}{1 - ck_1} + \frac{(k_2)_x}{1 - ck_2} = 0 .$$

If we set k_1 and k_2 at any umbilic p on S equal to the common value of the normal curvatures at p, then k_1 and k_2 become continuous functions over all of S. By (30) and the fact that umbilics on S are isolated, we conclude that

$$\log (1 - ck_1)(1 - ck_2)$$

is constant throughout S, so that

$$(1 - ck_1)(1 - ck_2) = \mathscr{C}$$

over S for some constant $\mathscr{C} > 0$.

LEMMA 14. *There is a smooth function $\mu \neq 0$ on S such that*

$$\Lambda = \mu(I - cII) ,$$

with $c \neq 0$ the constant in (31).

Proof of Lemma 14. Using (31) and the coordinates x, y of Lemma 12 anywhere on S^0, we get

$$\mathscr{C}(dx^2 + dy^2) = (k_1 - k_2)\{I - cII\} ,$$

which establishes $I - cII$ as a definite quadratic form proportional to Λ throughout S^0. Since $I - cII$ and Λ are each smooth quadratic forms on S, and since umbilics on S are isolated, the proof will be complete if we can show that $I - cII$ is definite at any umbilic p on S. But at such a p,

$$I - cII = I(1 - ck) ,$$

where $k = k_1 = k_2$, and (31) yields

$$k = \frac{1 \pm \sqrt{\mathscr{C}}}{c} .$$

Thus, at p,

$$I - cII = \begin{cases} \sqrt{\mathscr{C}}\, I, & \text{if } ck = 1 - \sqrt{\mathscr{C}} \\ -\sqrt{\mathscr{C}}\, I, & \text{if } ck = 1 + \sqrt{\mathscr{C}} . \end{cases}$$

In either case, $I - cII$ is definite since $\mathscr{C} > 0$. We have thus shown that (with the appropriate choice of sign)

$$R_A = R_{\pm(I-cII)} .$$

REMARK 6. In [11], J. Wolf studied surfaces which for some fixed constants α, β and γ satisfy

$$\alpha + \beta H + \gamma K = 0 . \tag{32}$$

We thus have an S of the type Wolf studied with (31′) yielding $\alpha = 1 - \mathscr{C}$, $\beta = -2c$ and $\gamma = c^2$. The conclusion in [11] was that for an S satisfying (32), the quadratic form $\tilde{A} = \alpha I + \beta II + \gamma III$ with $III = 2HII - KI$ must be flat wherever nondegenerate. For our S a brief computation shows that $\tilde{A}$ is everywhere degenerate.

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