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A NOTE ON p-SPACES AND MOORE SPACES

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In this note we investigate p-spaces and their relationship to Moore spaces. Specifically, it is shown that among p-spaces Moore spaces are equivalent to semi-metric spaces and spaces with a σ-discrete network. A certain class of p-spaces, called strict p-spaces, is given an internal characterization and this is used to show that a pointwise paracompact p-space with a point-countable base is a pointwise paracompact Moore space.

1. Developable p-spaces. Let us first discuss some of the definitions and basic concepts which will be used throughout this paper. Unless otherwise stated, all topological spaces are assumed to be T_2 and regular. The set of positive integers will be denoted by Z^+.

A sequence $\{\mathscr{U}_n\}_1^\infty$ of open covers of a space X is called a *development* for X if for any $x \in X$ and any open neighborhood O of x, there is an integer $n \in Z^+$ such that $St(x, \mathscr{U}_n) = \cup\{U \in \mathscr{U}_n : x \in U\} \subset O$. A regular developable space is a *Moore space*.

By Arhangel'skiî [1], a completely regular space X is called a *p-space* (plumed space) if in its Stone-Cech compactification $\beta(X)$ there is a sequence of families $\{\gamma_n\}_1^\infty$, where each γ_n is a collection of sets, open in $\beta(X)$, which covers X and satisfies: For each $x \in X$, $\bigcap_{n=1}^\infty St(x, \gamma_n) \subseteq X$. The sequence $\{\gamma_n\}_1^\infty$ is called a *pluming* for X in $\beta(X)$. A space X is called a *strict p-space* if it has a pluming $\{\gamma_n\}_1^\infty$ with the following additional property: For any $x \in X$ and any $n \in Z^+$ there is $n(x) \in Z^+$ such that $\overline{St(x, \gamma_{n(x)})} \subseteq St(x, \gamma_n)$. In this case we call $\{\gamma_n\}_1^\infty$ a *strict pluming*. The class of p-spaces includes all metric spaces, locally compact spaces and completely regular Moore spaces (see [1], [2]).

A collection $\mathscr{P}$ of subsets of a space X is called a *network* for X if for any open set $O \subseteq X$ and $x \in O$ there is a set $P \in \mathscr{P}$ such that $x \in P \subseteq O$.

Let X be a topological space and d a real valued nonnegative function defined on $X \times X$ which satisfies: For $x, y \in X$

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$.

The function d is called a *symmetric* [2] for the topology on X provided:

(3) $A \subseteq X$ is closed in X if and only if $d(x, A) = \inf\{d(x, z) : z \in A\} > 0$ for any $x \in X - A$.

The function d is called a *semi-metric* [9] for X provided:

(3′) For $A \subseteq X$, $x \in \bar{A}$ if and only if $d(x, A) = 0$.

It is clear that a semi-metric space is a symmetric space and it can easily be shown that a symmetric space X is a semi-metric space if and only if X is first countable. In [2] there is an example of a symmetric space which is not semi-metrizable. Aso, for later use, we note that open or closed subsets of symmetrizable space are symmetrizable and that compact symmetrizable spaces are metrizable [2].

In [2] Arhangel'skiî defines a space X to be *σ-paracompact* if for any open covering $\mathscr{U}$ of X there is a sequence $\{\mathscr{U}_n\}_1^\infty$ of open covers of X such that for any $x \in X$ there is $n(x) \in Z^+$ and some $U \in \mathscr{U}$ with $St(x, \mathscr{U}_{n(x)}) \subseteqq U$. Later, we give sufficient conditions for a space to be σ-paracompact and show that this property is enjoyed by a large class of topological spaces.

The following theorem, which is the main result of the section, answers some questions posed by Arhangel'skiî in [2].

THEOREM 1.1. *For a completely regular space X the following are equivalent*:

(a) *X has a development.*

(b) *X is a p-space with a σ-discrete network.*

(c) *X is a semi-metrizable p-space.*

(d) *X is a symmetrizable p-space.*

The following propositions are used to assist in the proof of Theorem 1.1.

PROPOSITION 1.2. *If X is a topological space with the property that every open cover of X has a σ-discrete refinement, then X is σ-paracompact.*

Proof. Let $\mathscr{U}$ be any open cover of X and suppose that $\mathscr{P} = \bigcup_{n=1}^\infty \mathscr{P}_n$ is a σ-discrete refinement of $\mathscr{U}$ with each $\mathscr{P}_n$ a discrete collection. We may assume the sets in $\mathscr{P}$ are closed. For each $P \in \mathscr{P}$ pick a set $U(P) \in \mathscr{U}$ which contains P. For each $n \in Z^+$ we define an open cover $\mathscr{U}_n$ as follows: If $x \in X$ and $x \in P$ for some $P \in \mathscr{P}_n$, define $U_n(x) = U(P) \cap [X - \cup\{P' \colon P' \in \mathscr{P}_n, x \notin P'\}]$. If $x \notin \cup\{P \colon P \in \mathscr{P}_n\}$, let $U_n(x) = X - \cup\{P \colon P \in \mathscr{P}_n\}$. Then $\mathscr{U}_n = \{U_n(x) \colon x \in X\}$ is an open cover of X for each $n \in Z^+$. It is clear that $x \in P \in \mathscr{P}_n$ implies that $St(x, \mathscr{U}_n) \subseteqq St(P, \mathscr{U}_n) \subseteqq U(P)$. Since every $x \in X$ is is some element of $\mathscr{P}$, it follows that X is σ-paracompact.

McAuley [7] has shown that in a semi-metric space every open cover has a σ-discrete closed refinement.

COROLLARY 1.3. *Any semi-metric space is σ-paracompact.*

The next corollary follows immediately from Proposition 1.2 and the definition of a σ-discrete network.

COROLLARY 1.4. *Any topological space X with a σ-discrete network is σ-paracompact.*

LEMMA 1.5. *If X is a p-space and $x \in X$ such that $\{x\}$ is a G_δ set in X then x has a countable neighborhood base.*

Proof. Let $\{\gamma_n\}_1^\infty$ be a pluming for x in $\beta(X)$ and suppose $\{x\} = \bigcap_{n=1}^\infty G_n$, where $G_n \subseteqq X$ is open in X. For each $n \in Z^+$ there is a set G'_n open in $\beta(X)$ such that $G_n = X \cap G'_n$. It follows that $\{x\} = [\bigcap_{n=1}^\infty G'_n] \cap [\bigcap_{n=1}^\infty St(x, \gamma_n)]$. Thus $\{x\}$ is a G_δ set in $\beta(X)$ and must have a countable neighborhood base in $\beta(X)$ since $\beta(X)$ is compact. This neighborhood base relative to X will give a neighborhood base for x in X.

PROPOSITION 1.6. *A symmetrizable p-space X is semi-metrizable.*

Proof. By our previous remarks it suffices to show that X satisfies the first axiom of countability and for this it is enough to show that each point $x \in X$ is a G_δ subset of $\beta(X)$. Assume $\{\gamma_n\}_1^\infty$ is a pluming for X in $\beta(X)$; then for each $n \in Z^+$ there is a $\beta(X)$ open neighborhood of x, say $U_n(x)$, such that $\overline{U_n(x)} \subseteqq St(x, \gamma_n)$. Let $C = \bigcap_{n=1}^\infty \overline{U_n(x)} \subseteqq \bigcap_{n=1}^\infty St(x, \gamma_n) \subseteqq X$. Then C is a closed compact set in X since it is compact in $\beta(X)$. Recall that a closed subset of a symmetrizable space is symmetrizable and a compact symmetrizable space is metrizable. Thus C is metrizable and there is a sequence $\{N_n(x)\}_1^\infty$ of open sets in $\beta(X)$ such that $\{C \cap N_n(x) \colon n \in Z^+\}$ is a neighborhood base at x relative to C. Clearly $\{x\} = [\bigcap_{n=1}^\infty U_n(x)] \cap [\bigcap_{n=1}^\infty N_n(x)]$; so the proposition is proved.

PROPOSITION 1.7. *A p-space X with a σ-discrete network is semi-metrizable.*

Proof. Since X has a σ-discrete network it is clear that all closed sets, and in particular singleton sets, are G_δ sets. By Lemma 1.5 X is a first countable space with a σ-discrete network and consequently is semi-metrizable by a theorem of Arhangel'skiĭ [2].

PROPOSITION 1.8. *A space X with a development $\{\mathscr{U}_n\}_1^\infty$ has a σ-discrete network.*

Proof. A developable space is semi-metrizable and thus each $\mathscr{U}_n$ has a σ-discrete closed refinement $\mathscr{B}_n$. It is easy to show that $\bigcup_{n=1}^\infty \mathscr{B}_n$ is a σ-discrete network for X.

Proof of Theorem 1.1. Since a completely regular Moore space is a p-space, Propositions 1.7 and 1.8 show that (a) implies (b) and (b) implies (c). That (c) implies (d) is obvious, so all that remains is to show that (d) implies (a). If we assume condition (d), then X has a semi-metric d by Proposition 1.6 and is also σ-paracompact by Corollary 1.3. Let $\{\gamma_n\}_1^\infty$ be a pluming for X in $\beta(X)$. For each $x \in X$ let $S'_n(x)$ be a neighborhood in $\beta(X)$ such that $S'_n(x) \cap X = S_n(x)$, where $S_n(x) = \{y \in X: d(x, y) < 1/n\}$. Keep in mind that the collection $\{S_n(x): n \in Z^+\}$ is a base (not necessarily open) for the neighborhood system at x in X. For each $x \in X$ let $U_n(x)$ be a neighborhood of x in $\beta(X)$ such that $\overline{U_n(x)} \subseteqq S'_n(x)$ and such that the family $\{\overline{U_n(x)}: x \in X\}$ refines γ_n. Let $\mathscr{U}(n) = \{U_n(x) \cap X: x \in X\}$. Now, since X is σ-paracompact, for each $n \in Z^+$ there is a sequence $\{\mathscr{U}_m(n)\}_{m=1}^\infty$ of open covers of X that has the following property: For $x \in X$, there is $m(x) \in Z^+$ and $U \in \mathscr{U}(n)$ such that $St(x, \mathscr{U}_{m(x)}(n)) \subseteqq U$. We may assume $\mathscr{U}_{m+1}(n)$ refines $\mathscr{U}_m(n)$ for each $m \in Z^+$. Finally, for $n \in Z^+$ let $\mathscr{G}_n$ be an open cover of X such that $\mathscr{G}_n$ refines each $\mathscr{U}_s(t)$ for $s \leqq n$, $t \leqq n$ and $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$. We show that the sequence $\{\mathscr{G}_n\}_{n=1}^\infty$ is a development for X. If $x \in X$ is fixed and k is any positive integer, than there is some $x_k \in X$ and $n_k \in Z^+$, $n_k \geqq k$ such that $St(x, \mathscr{U}_{n_k}(k)) \subseteqq U_k(x_k) \cap X$. So $St(x, \mathscr{G}_{n_k}) \subseteqq St(x, \mathscr{U}_{n_k}(k)) \subseteqq U_k(x_k) \cap X$. We may assume $n_{k+1} > n_k$ so that $St(x, \mathscr{G}_{n_{k+1}}) \subseteqq St(x, \mathscr{G}_{n_k})$. Suppose O is any neighborhood of x, open in X, and O' is a set open in $\beta(X)$ such that $O = X \cap O'$. If $\bigcap_{k=1}^m \overline{U_k(x_k)}$ is not contained in O' for any $m \in Z^+$, then

$$\left\{\bigcap_{k=1}^m \overline{U_k(x_k)} - O'\right\}_{m=1}^\infty$$

is a decreasing sequence of nonempty closed sets in $\beta(X)$ and hence $\bigcap_{k=1}^\infty \overline{U_k(x_k)} - O' \neq \varnothing$. Now

$$\bigcap_{k=1}^\infty \overline{U_k(x_k)} \subseteqq \left[\bigcap_{k=1}^\infty St(x, \gamma_k)\right] \cap \left[\bigcap_{k=1}^\infty S'_k(x_k)\right] \subseteqq X \cap \left[\bigcap_{k=1}^\infty S'_k(x_k)\right] = \bigcap_{k=1}^\infty S_k(x_k) \, .$$

Hence $y \in \bigcap_{k=1}^\infty \overline{U_k(x_k)} - O'$ implies that $y \in \bigcap_{k=1}^\infty S_k(x_k)$ and so it follows that $\{x_k\}_1^\infty \to y$. But also $\{x_k\}_1^\infty \to x$; so $x = y$ which is a contradiction. So there is a positive integer m such that $\bigcap_{k=1}^m \overline{U_k(x_k)} \subseteqq O'$. It follows that $St(x, \mathscr{G}_{n_m}) \subseteqq \bigcap_{k=1}^m St(x, \mathscr{G}_{n_k}) \subseteqq [\bigcap_{k=1}^m U_k(x_k)] \cap X \subseteqq O' \cap X = O$. Hence $x \in St(x, \mathscr{G}_{n_m}) \subseteqq O$ and the theorem is proved.

COROLLARY 1.9 (*Arhangel'skiĭ*). *A collectionwise normal p-space with a symmetric* (*or with a σ-discrete network*) *is metrizable.*

2. A characterization of strict p-spaces.

LEMMA 2.1. *A pointwise paracompact p-space X has a pluming*

$\{\gamma_n\}_1^\infty$ *in* $\beta(X)$ *which satisfies*:

(a) *Each* γ_n *is point-finite at points in* X;

(b) *For each* $x \in X$ *and* $n \in Z^+$, $\overline{St(x, \gamma_{n+1})} \subseteqq St(x, \gamma_n)$.

The proof of Lemma 2.1 is straightforward and is omitted.

The next theorem gives an internal characterization of strict p-spaces; that is, strict p-spaces are defined without use of the compactification $\beta(X)$. In some cases this characterization has proved to be more useful than the original definition.

THEOREM 2.2. *A completely regular space* X *is a strict* p*-space if and only if there is a sequence* $\{\mathscr{G}_n\}_{n=1}^\infty$ *of open covers of* X *satisfying*:

(a) $P_x = \bigcap_{n=1}^\infty St(x, \mathscr{G}_n)$ *is a compact set for each* $x \in X$;

(b) *The family* $\{St(x, \mathscr{G}_n): n \in Z^+\}$ *is a neighborhood base for the set* P_x.

Proof. If X is a strict p-space, there is a strict pluming $\{\gamma_n\}_1^\infty$ for X in $\beta(X)$ where we may assume that γ_{n+1} is a refinement of γ_n. Let $P_x = \bigcap_{n=1}^\infty \overline{St(x, \gamma_n)} = \bigcap_{n=1}^\infty St(x, \gamma_n)$ and $\mathscr{G}_n = \{G \cap X: G \in \gamma_n\}$. Clearly, P_x is a compact subset of X and $P_x = \bigcap_{n=1}^\infty St(x, \mathscr{G}_n)$. To show that $\{St(x, \mathscr{G}_n): n \in Z^+\}$ is a neighborhood base for the set P_x, let U be any open set in X which contains P_x and let U' be an open set in $\beta(X)$ where $U = X \cap U'$. Now if the set $\bigcap_{k=1}^n \overline{St(x, \gamma_k)} - U'$, which is closed in $\beta(X)$, is not empty for any $n \in Z^+$ we have $\bigcap_{k=1}^\infty \overline{St(x, \gamma_k)} - U' \neq \varnothing$. This is impossible; hence there is a positive integer n such that $\bigcap_{k=1}^n \overline{St(x, \gamma_k)} \subseteqq U'$. Since $St(x, \mathscr{G}_n) \subseteqq St(x, \gamma_n) \cap X \subseteqq [\bigcap_{k=1}^n St(x, \gamma_k)] \cap X$ it follows that $P_x \subseteqq St(x, \mathscr{G}_n) \subseteqq U$.

Conversely, suppose $\{\mathscr{G}_n\}_{n=1}^\infty$ is a sequence of open covers of X which satisfies (a) and (b). Let γ_n be the collection of all sets G' open in $\beta(X)$ such that $G' \cap X \in \mathscr{G}_n$; we will show that $\{\gamma_n\}_{n=1}^\infty$ is a strict pluming for X in $\beta(X)$. Let $x \in X$, $y \in \beta(X) - X$, $n \in Z^+$ and O be an open set in $\beta(X)$ with $P_x \subseteqq O \subseteqq \bar{O} \subseteqq St(x, \gamma_n) \cap [B(X) - \{y\}]$. By assumption there is an integer $n' \in Z^+$ such that $P_x \subseteqq St(x, \mathscr{G}_{n'}) \subseteqq \bar{O} \cap X$. But $St(x, \mathscr{G}_{n'}) = St(x, \gamma_{n'}) \cap X$ which implies that $St(x, \gamma_{n'}) - O$ is an open subset of $\beta(X) - X$ and therefore must be empty. So $St(x, \gamma_{n'}) \subseteqq \bar{O} \subseteqq \beta(X) - \{y\}$ and y was an arbitrary element of $\beta(X) - X$. Hence $\bigcap_{n=1}^\infty St(x, \gamma_n) \subseteqq X$ and $\{\gamma_n\}_{n=1}^\infty$ is a pluming for X. We also have $St(x, \gamma_{n'}) \subseteqq \bar{O} \subseteqq St(x, \gamma_n)$; so $\overline{St(x, \gamma_{n'})} \subseteqq St(x, \gamma_n)$. Thus $\{\gamma_n\}_{n=1}^\infty$ is a strict pluming for X.

REMARK 2.3. (a) Notice that if the sets P_x, as in Theorem 2.2, are singleton sets, then sequence $\{\mathscr{G}_n\}_1^\infty$ is actually a development for X. (b) If $\{\mathscr{G}_n\}_{n=1}^\infty$ is a sequence of open covers satisfying (a) and (b) in Theorem 2.2, the following is easily verified: If $x \in X$ and $\{x_n\}_{n=1}^\infty$ is any sequence of points in X such that $x_n \in \bigcap_{i=1}^n St(x, \mathscr{G}_i)$ for each $n \in Z^+$, then $\{x_n\}_{n=1}^\infty$ has a cluster point in $P_x = \bigcap_{n=1}^\infty St(x, \mathscr{G}_n)$.

PROPOSITION 2.4. *If X is a pointwise paracompact p-space, then there is a sequence $\{\mathscr{G}_n\}_1^\infty$ of point-finite open covers of X satisfying*:

(a) *$P_x = \bigcap_{n=1}^\infty St(x, \mathscr{G}_n)$ is a compact set for each $x \in X$;*

(b) *The family $\{St(x, \mathscr{G}_n): n \in Z^+\}$ is a neighborhood base for the set P_x.*

Proposition 2.4 follows from Lemma 2.1 and the proof of Theorem 2.2.

LEMMA 2.5. *If $\mathscr{B}$ is a point-countable collection of subsets of a space X and $A \subseteq X$, then the family of all minimal finite covers of A with elements from $\mathscr{B}$ is countable.*

LEMMA 2.6. *A topological space X is semi-metrizable if and only if at each point $x \in X$ there is a decreasing open neighborhood base $\{U_n(x)\}_{n=1}^\infty$ such that the following is true: If $\{x_n\}_1^\infty$ is any sequence of points in X and $y \in X$ such that $y \in U_n(x_n)$ for each $n \in Z^+$, then $\{x_n\}_1^\infty \rightarrow y$.*

Lemmas 2.5 and 2.6 are used in the proof of Theorem 2.7. Lemma 2.5 can be found in [4] and Lemma 2.6 in [6]. Theorem 2.7 generalizes a result due to Filippov [4].

THEOREM 2.7. *A pointwise paracompact p-space X with a point countable base $\mathscr{B}$ is a pointwise paracompact Moore space.*

Proof. First we show that X has a σ-point-finite base for its topology. By Proposition 2.4 there is a sequence $\{\mathscr{G}_n\}_1^\infty$ of point-finite open covers of X such that for any $x \in X$, $P_x = \bigcap_{n=1}^\infty St(x, \mathscr{G}_n)$ is compact and the family $\{St(x, \mathscr{G}_n): n \in Z^+\}$ is a neighborbood base for P_x. Let $n \in Z^+$ and $G \in \mathscr{G}_n$. By Lemma 2.5 there are at most a countable number of minimal finite covers of G by elements of $\mathscr{B}$, say $G(1, n)$, $G(2, n), G(3, n), \cdots$ (if they exist). For $k \in Z^+$ let $\mathscr{U}_{k,n} = \{B \cap G: G \in \mathscr{G}_n$ and $B \in G(k, n)\}$. For fixed k and $n \in Z^+$, $\mathscr{U}_{k,n}$ is point-finite since each $x \in X$ is in at most a finite number of $G \in \mathscr{G}_n$ and in only a finite number of elements from $G(k, n)$. To show that $\mathscr{U} = \bigcup_{k=1}^\infty \bigcup_{n=1}^\infty \mathscr{U}_{k,n}$ is a base for the topology, let O be an arbitrary open neighborhood of a point $x \in X$. There is a $B \in \mathscr{B}$ such that $x \in B \subseteqq O$ and since P_x is compact, we can find a finite cover $\{B_1, B_2, \cdots, B_k\}$ of P_x with elements from $\mathscr{B}$ such that $B = B_1$ and B is the only element in the cover which contains x. Because $\bigcup_{i=1}^{k} B_i$ is an open neighborhood of P_x, there is an integer $m \in Z^+$ such that $P_x \subseteqq St(x, \mathscr{G}_m) \subseteqq \bigcup_{i=1}^{k} B_i$. If G is any element of $\mathscr{G}_m$ which contains x, there is a minimal subcover from $\{B_1, B_2, \cdots, B_k\}$ which covers G, say $G(j, m)$ for some $j \in Z^+$. But B must be an element of $G(j, m)$; hence $B \cap G \in \mathscr{U}_{j,m}$ and $x \in B \cap G \subseteqq O$.

To complete the proof, we use Lemma 2.6 to show that X is semi-metrizable and apply Theorem 1.1. Assume $X \in \mathscr{U}_{k,m}$ for all $k, m \in Z^+$. Let $x \in X$ and $n \in Z^+$. Define $U'_n(x) = \cap\{U \in \mathscr{U}_{k,m}: x \in U, k \leqq n, m \leqq n\}$ and $U_n(x) = U'_n(x) \cap [\bigcap_{k=1}^{n} St(x, \mathscr{G}_k)]$. Suppose for some $x \in X$ that there is a sequence $\{x_n\}_1^\infty$ of points in X such that $x \in U_n(x_n)$ for each $n \in Z^+$. Then $x \in \bigcap_{i=1}^{n} St(x_n, \mathscr{G}_i)$ for each $n \in Z^+$ which implies that $x_n \in \bigcap_{i=1}^{n} St(x, \mathscr{G}_i)$. By Remark 2.3b $\{x_n\}_1^\infty$ has a cluster point $y \in \bigcap_{n=1}^{\infty} St(x, \mathscr{G}_n)$. If $y \neq x$, then there exists $m \in Z^+$ such that $x \notin U'_m(y)$. But also, since y is a cluster point of $\{x_n\}$, there exists a positive integer $m_1 \geqq m$ such that $x_{m_1} \in U'_m(y)$. Thus $U'_m(x_{m_1}) \subseteqq U'_m(y)$ and $x \in U'_{m_1}(x_{m_1}) \subseteqq U'_m(x_{m1}) \subseteqq U'_m(y)$ which is a contradiction. Therefore x is the only cluster point of $\{x_n\}_1^\infty$. Let $\{x_{n_k}\}_{k=1}^\infty$ be any subsequence of $\{x_n\}^\infty$. Since $x \in U_{n_k}(x_{n_k}) \subseteqq \bigcap_{i=1}^{n_k} St(x_{n_k}, \mathscr{G}_i) \subseteqq \bigcap_{i=1}^{k} St(x_{n_k}, \mathscr{G}_i)$, it follows that $x_{n_k} \in \bigcap_{i=1}^{k} St(x, \mathscr{G}_i)$. So $\{x_{n_k}\}_{k=1}^\infty$ has a cluster point by Remark 2.3b which from above must be x. Thus $\{x_n\}_1^\infty \to x$.

We have satisfied the conditions of Lemma 2.6; so X is semi-metrizable and hence developable.

Addendum. It has recently come to the author's attention that Theorem 1.1, (a) $\Leftrightarrow$ (c), and Theorem 2.7 were announced in [10] by Creede and Heath.

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