PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 2, 1969

ON COMMUTATIVE RINGS OVER WHICH THE SINGULAR SUBMODULE IS A DIRECT SUMMAND FOR EVERY MODULE

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A commutative ring R with 1 over which the singular submodule is a direct summand for every module, is a semi-hereditary ring with finitely many large ideals. A commutative semi-simple (with d.c.c.) ring is characterized by the property that every semi-simple module is injective.

In this note we continue our investigation of the commutative non-singular rings over which for any module M, the singular submodule $Z(M)$ is a direct summand [1]. As in [1] we say that such a ring has SP. Throughout this paper a ring R is commutative with identity 1 and all modules are unitary. A ring is regular (in the sense of Von Neumann) if every finitely generated right ideal of R is generated by an idempotent; semi-simple means semi-simple with d.c.c.. Notation and terminology here is as in [1].

In [1] we established the following characterization of rings with SP, included here for easy reference:

THEOREM 1. *For a ring R the following are equivalent*:

(a) *R has SP*

(b) *R is regular and has BSP* [1].

(c) *$Z(R) = (0)$ and for every large ideal I or R, the ring R/I is semi-simple.*

(d) *Every R-module M with $Z(M) = M$ is R-injective.*

In particular if R has SP, then R is hereditary.

We shall need the following corollaries of this theorem:

COROLLARY 1.1. *Every homomorphic image of a ring R with SP, has SP.*

Proof. Let $S = R/I$ for some ideal $I(\neq (0), R)$ of R; S is regular since R is and thus $Z(S) = (0)$. A large ideal A of S is of the form J/I where J is a large ideal of R containing I. Thus it follows from (c) that S/A is semi-simple since $S/A \cong R/J$. Now S has SP since it satisfies (c).

COROLLARY 1.2. *Every singular module over a ring R with SP*

is semi-simple.

Proof. Every cyclic singular module is semi-simple by (c).

For any R-module M we denote by $\mathrm{So}_R(M)$ (or $\mathrm{So}(M)$ if no ambiguity arises) the socle of M, that is the module sum of the simple submodules of M.

THEOREM 2. *Over a ring R with SP the socle,* $\mathrm{So}(M)$, *of every nonzero R-module M is large in M.*

Proof. By definition of essential extension, it suffices to show that every nonzero cyclic module R/I has nonzero socle.

We have:

LEMMA 3. *Let R be a ring with SP and $\{e_n: n \in A\}$ a countable set of orthogonal idempotents of R such that e_nR contains a proper $(\neq e_nR)$ large R-submodule I_n, for each $n \in A$. Then, the set $\{e_n: n \in A\}$ is finite.*

Proof. For each $n \in A$ the module $e_nR/I_n = S_n \neq (0)$ is singular. It follows that $\bigoplus_{n \in A} S_n$ is singular and hence injective by Theorem 1 (d). Let $g: \bigoplus_{n \in A} e_nR \rightarrow \bigoplus_{n \in A} S_n$ be the R-homomorphism defined by $g \mid e_nR: e_nR \rightarrow e_nR/I_n$, the natural R-epimorphism. Since $\bigoplus_{n \in A} e_nR \subseteqq R$ and $\bigoplus_{n \in A} S_n$ is injective, there is extension $g^*: R \rightarrow \bigoplus S_n$ of g and since $\mathrm{Im}\, g^*$ is clearly contained in only a finite number of the S_n, it follows that they are finitely many. Hence $\{e_n: n \in A\}$ is a finite set.

To return to the proof of Theorem 2, we note that Corollary 1.1 and Lemma 3 imply that if R is a ring with SP and $\{e_n: n \in A\}$ is an infinite set of orthogonal idempotents of R/I, then all but finitely many of the ideals $e_n(R/I)$ are semi-simple. If R/I has no infinite set of orthogonal idempotents, then R/I itself is semi-simple.

To establish the characterizations announced at the beginning of this paper we need the well known characterizations (e.g. [2]) of a commutative regular ring contained in Theorem 4 below. For each module M, $J(M)$ denotes the radical of M.

THEOREM 4. *For a commutative ring R the following are equivalent:*

(a) *R is regular.*

(b) *Every simple R-module is injective.*

(c) $J(M) = (0)$ *for every R-module M.*

(d) *Every ideal of R is the intersection of the maximal ideals that contain it.*

We can now complete the characterization of rings with SP, partially established in [1] as Theorem 2.9.

THEOREM 5. *For a ring R the following are equivalent*:

(a) *R has SP.*

(b) *$R/\mathrm{So}(R)$ is a semi-simple ring and R is regular.*

(c) *R is semi-hereditary and has only a finite number of large ideals.*

Proof. (a) implies (b). A consequence of parts (b) and (c) of Theorem 1 and Theorem 2.

(b) implies (c). R is certainly semi-hereditary since it is regular. Furthermore since every large ideal of R contains the socle of R, $R/\mathrm{So}(R)$ is semi-simple implies that R has only a finite number of maximal large ideals. Since every ideal containing a large ideal is itself large, the assertion that R has only a finite number of large ideals follows now from (d) of Theorem 4.

(c) implies (a). Theorem 2.9 [2].

The proof of Theorem 5 is now complete.

We close with the following characterization of commutative semi-simple rings:

THEOREM 6. *For a commutative ring R the following are equivalent*:

(a) *R is semi-simple.*

(b) *Every semi-simple R-module is injective.*

Proof. (b) implies (a). From (b) and part (b) of Theorem 4 it follows that R is regular. It is, hence, sufficient to show that R contains no infinite sets of orthogonal idempotents. Thus let $\{e_n: n \in A\}$ be any countable set of orthogonal idempotents. It follows from part (c) of Theorem 4 that each e_nR contains a maximal R-submodule I_n and if we let $S_n = e_nR/I_n$ then $S = \bigoplus_{n \in A} S_n$ is a semi-simple R-module. From (b) S is injective and an argument similar to the one used to prove Lemma 3, gives now that the set $\{e_n: n \in A\}$ is finite. Hence R contains no infinite sets of orthogonal idempotents.

We do not know whether Theorem 6 remains true if R is not assumed commutative. In this direction it can be shown that if (b) is assumed for right modules then R is finite dimensional on the right (i.e., R contains no infinite direct sum of nonzero right ideals).

REFERENCES

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Received February 18, 1969.

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