ON \((m - n)\) PRODUCTS OF BOOLEAN ALGEBRAS

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This discussion begins with the problem of whether or not all \((m - n)\) products of an indexed set \(\{\mathcal{U}_t\}_{t \in T}\) of Boolean algebras can be obtained as \(m\)-extensions of a particular algebra \(\mathcal{F}_n^*\). The construction of \(\mathcal{F}_n^*\) is similar to the construction of the Boolean product of \(\{\mathcal{U}_t\}_{t \in T}\); however the \(\mathcal{U}_t\) are embedded in \(\mathcal{F}_n^*\) in such a way that their images are \(n\)-independent. If there is a cardinal number \(n'\), satisfying \(n < n' \leq m\), then \((m - n')\) products are not obtainable in this manner. For the case \(n = m\) an example shows the answer to be negative. It is explained how the class of \(m\)-extensions of \(\mathcal{F}_n^*\) is situated in the class of all \((m - n)\) products of \(\{\mathcal{U}_t\}_{t \in T}\).

A set of \(m\)-representable Boolean algebras is given for which the minimal \((m - n)\) product is not \(m\)-representable and for which there is no smallest \((m - n)\) product.

These problems have been proposed by R. Sikorski (see [2]). Concerning \(\{\mathcal{U}_t\}_{t \in T}\), it is assumed throughout that each of these algebras has at least four elements. \(m\) and \(n\) will always denote infinite cardinals with \(n \leq m\). All definitions are taken from [2]. An \(m\)-homomorphism is a homomorphism that is conditionally \(m\)-complete. We denote the class of \((m - n)\) products of \(\{\mathcal{U}_t\}_{t \in T}\) by \(P_n\) and the class of \((m - 0)\) products by \(P\). Let \(\{\{i_t\}_{t \in T}, \mathcal{B}\}\) and \(\{\{j_t\}_{t \in T}, \mathcal{C}\}\) be elements of \(P\). We say that

\[\{\{i_t\}_{t \in T}, \mathcal{B}\} \leq \{\{j_t\}_{t \in T}, \mathcal{C}\}\]

provided there is an \(m\)-homomorphism \(h\) from \(\mathcal{C}\) onto \(\mathcal{B}\) such that \(h \circ j_t = i_t\) for \(t \in T\). The relation \(\leq\) is a quasi-ordering of \(P\). Two \((m - 0)\) products are isomorphic if each is \(\leq\) to the other.

The particular product, \(\{\{g^*_t\}_{t \in T}, \mathcal{F}_n^*\}\) of \(\{\mathcal{U}_t\}_{t \in T}\) mentioned above is defined as follows. For each \(t \in T\) let \(X_t\) be the Stone space of \(\mathcal{U}_t\) and let \(g_t\) be an isomorphism from \(\mathcal{U}_t\) onto the field \(\mathcal{F}_t^*\) of all open and closed subsets of \(X_t\). Let \(X\) be the Cartesian product of the sets \(X_t\), and for each \(t \in T\) and each \(b \in \mathcal{U}_t\), set

\[g^*_t(b) = \{x \in X: x(t) \in g_t(b)\} .\]

Let \(G_n^*\) be the set of all subsets \(a\) of \(X\) which satisfy the following condition:

\[a = \bigcap_{t \in S} g^*_t(b_t)\text{ where } b_t \in \mathcal{U}_t, S \subseteq T \text{ and } \overline{S} \subseteq n .\]

Finally, let \(\mathcal{F}_n^*\) be the field of subsets of \(X\) which is generated by \(G_n^*\).
is a base for the \( n \)-topology on \( X \). \( g_t^* \) is a complete isomorphism from \( \mathcal{A}_t \) into \( \mathcal{F}_n^* \). The set \( \{g_t^*(\mathcal{A}_t)\} \), of subalgebras, is \( n \)-independent.

A Boolean \((m - n)\) product \( \{\mathcal{B}_t\}_{t \in T} \) is said to belong to \( \mathcal{E}_n \) if and only if there is an \( m \)-isomorphism \( h \) (from \( \mathcal{F}_n^* \) into \( \mathcal{B}_t \)) such that \( \{h, \mathcal{B}_t\} \) is an \( m \)-extension of \( \mathcal{F}_n^* \) and for each \( t \in T \) \( h \circ g_t^* = i_t \).

For every \( m \)-extension \( \{h, \mathcal{B}_t\} \) of \( \mathcal{F}_n^* \), \( \{(h \circ g_t^*)_t\}_{t \in T} \) \( \mathcal{E}_n \). Clearly \( \mathcal{E}_n \subseteq \mathcal{P}_n \) and \( \mathcal{E}_n \) is not empty. \( m \)-extensions of \( \mathcal{F}_n^* \) seem to provide the most natural examples of Boolean \((m - n)\) products.

1. **Lemma 1.1.** Let \( \{\mathcal{B}_t\}_{t \in T} \) be an \( n \)-independent set of subalgebras of a Boolean algebra \( \mathcal{A} \) and let \( S \) and \( S' \) be subsets of \( T \) with \( \overline{S} \leq n \) and \( \overline{S'} \leq n \). For each \( t \) let \( a_t \) and \( b_t \) be nonzero elements of \( \mathcal{B}_t \). Then

\[
\begin{align*}
(1) & \quad \prod_{t \in S} a_t \leq \prod_{t \in S} b_t \text{ if and only if } a_t \leq b_t \text{ for each } t \in S; \\
(2) & \quad \prod_{t \in S} a_t = \prod_{t \in S} b_t \text{ implies that } a_t = b_t \text{ for } t \in S \cap S', \quad a_t = 1 \text{ for } t \in S - S', \quad \text{and } b_t = 1 \text{ for } t \in S' - S.
\end{align*}
\]

**Proof.** (i) Assume that for some \( t_0 \in S \), \( a_{t_0} \nleq b_{t_0} \). Define

\[ C_t = \begin{cases} a_t & \text{if } t \in S \text{ and } t \neq t_0, \\ a_{t_0} \cdot (-b_{t_0}) & \text{if } t = t_0. \end{cases} \]

Set \( c = \prod_{t \in S} c_t \), and note that \( c \neq 0 \), \( c \leq \prod_{t \in S} a_t \), and \( c \cdot \prod_{t \in S} b_t = 0 \). The converse is clear.

To prove (ii) we define

\[
\begin{align*}
x_t = \begin{cases} a_t & \text{if } t \in S, \\ 1 & \text{if } t \in S' - S; \end{cases} \quad \text{and} \quad y_t = \begin{cases} b_t & \text{if } t \in S', \\ 1 & \text{if } t \in S - S'. \end{cases}
\end{align*}
\]

Now

\[
\prod_{t \in S \cup S'} x_t = \prod_{t \in S} a_t = \prod_{t \in S} b_t = \prod_{t \in S \cup S'} y_t
\]

and (ii) follows from (i).

**Lemma 1.2.** Let \( \{\mathcal{B}_t\}_{t \in T} \) be an \( n \)-independent set of subalgebras of a Boolean algebra \( \mathcal{A} \). Let \( G \) be the set of all meets \( \prod_{t \in S} a_t \) such that \( S \subseteq T \), \( \overline{S} \leq n \), and for each \( t \in S \) \( a_t \) is a nonzero element of \( \mathcal{B}_t \). Assume further that \( G \) generates \( \mathcal{A} \). Then \( G \) is dense in \( \mathcal{A} \).

**Proof.** First note that for \( g, g' \in G \) either \( g \cdot g' = 0 \) or else \( g \cdot g' \in G \). Thus every nonzero element of \( \mathcal{A} \) is a finite join of elements of the form \( g \cdot \prod_{t \in S} (-g_t) \) with \( g, g_t \in G \) and \( k \) finite. (This notation is intended
to include the special cases $g$ and $-g$.) Now suppose $g \cdot \prod_{i<k} (-g_i) \neq 0,$ so that $g \not\subset \sum_{i<k} g_i$. We write a common form $g = \prod_{i \in S} a_i,$ and for each $i < k$, $g_i = \prod_{i \in S} a_{i,t}$ where $S \subseteq T$, $\bar{S} \leq n$, and for each $t \in S$ $a_i$ and $a_{i,t}$ are nonzero elements of $\mathcal{B}$. Since $k$ is finite every Boolean algebra is $(k - n)$-distributive (see [2], p. 62). We have

$$\prod_{i \in S} a_i = \sum_{i \leq k} \prod_{i \in S} a_{i,t} = \prod_{\phi \in S^k} \sum_{i \leq k} a_{i,\phi(i)}.$$  

(Here $S^k$ denotes the set of all functions from $k = \{0, 1, \ldots, k-1\}$ into $S$.) Choose $\phi \in S^k$ such that $\prod_{i \in S} a_i = \sum_{i \leq k} a_{i,\phi(i)}$. We have, for each $s \in \{\phi(i): i < k\}$, 

$$a_s = \sum_{\phi(i) = s} a_{i,\phi(i)}.$$  

Define

$$b_t = \begin{cases} a_t, & \text{if } t \in S - \{\phi(i): i < k\} \\ a_t - \sum_{\phi(i)=t} a_{i,\phi(i)}, & \text{if } t \in \{\phi(i): i < k\}. \end{cases}$$

Finally let $b = \prod_{i \in S} b_i$. Clearly $b \neq 0$, $b \in G$ and $b \leq g$. For each $t \in \{\phi(i): i < k\}$, $b \cdot \sum_{\phi(i)=t} a_{i,\phi(i)} = 0$, so that $b \cdot \sum_{i \leq k} a_{i,\phi(i)} = 0$. It follows that $b \cdot \sum_{i \leq k} g_i = 0$, hence $b \leq g \cdot \prod_{i \leq k} (-g_i)$.

**Corollary 1.3.** If $\bar{S} > n$, and for each $t \in S$, $a_i \neq 1$, then $\prod_{i \in S} a_i = 0$.

**Theorem 1.4.** Let $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in \mathcal{P}_n$. There is one and only one isomorphism $h_n$ from $\mathcal{F}_n$ into $\mathcal{B}$ which satisfies the following completeness condition:

$c)$ $h_n(\prod_{i \in S} g^*_i(a_i)) = \prod_{i \in S} i_t(a_i)$ whenever $S \subseteq T$, $\bar{S} \leq n$,

$$a_i \in \mathcal{A}_t$$

and $a_i \neq 0$.

**Proof.** Let $G$ be the set of all meets $\prod_{i \in S} i_t(a_i)$ such that $S \subseteq T$, $\bar{S} \leq n$, each $a_i \in \mathcal{A}_t$ and $a_i \neq 0$. Let $\mathcal{A}$ be the subalgebra of $\mathcal{B}$ which is generated by $G$. For $\prod_{i \in S} i_t(a_i) \in G$ it is clear that $\prod_{i \in S} i_t(a_i) = \prod_{i \in S} i_t(a_i)$. By Lemma 1.2 $G$ is dense in $\mathcal{A}$. Also $G_n$ is dense in $\mathcal{F}_n$. For $a \in G_n$ write $a = \bigcap_{i \in S} g^*_i(a_i) = \prod_{i \in S} g^*_i(a_i)$. Define $h(a) = \prod_{i \in S} i_t(a_i)$. It is easily seen, using Lemma 1.1, that

(i) $h$ is a one to one function from $G_n$ onto $G$;

(ii) for $a, b \in G_n$, $a \leq b$ if and only if $h(a) \leq h(b)$.

It follows (see [2], p. 37) that $h$ can be extended to an isomorphism $h_n$ from $\mathcal{F}_n$ onto $\mathcal{A}$. $h_n$ is uniquely determined by condition (c) because $G_n$ generates $\mathcal{F}_n$.

**Corollary 1.5.** The product $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in \mathcal{E}_n$ if and only if $h_n$ is $n$-complete.
Proof. Let \( \{ \{ i_t \}_{t \in T}, \mathcal{B} \} \in E_n \). There is an \( m \)-isomorphism \( f \) from \( \mathcal{F}_n^* \) into \( \mathcal{B} \) such that for each \( t \in T \), \( f \circ g_t^* = i_t \). \( f \) satisfies condition (c) so \( f = h_n \).

**Corollary 1.6.** Assume \( \tilde{T} > n \) and that \( m \geq n' > n \). Then \( P_n' \cap E_n \) is empty.

**Proof.** Let \( \{ \{ i_t \}_{t \in T}, \mathcal{B} \} \in P_n' \). Consider the isomorphism \( h_n \) from \( \mathcal{F}_n^* \) into \( \mathcal{B} \). Choose \( S \subseteq T \), \( \tilde{S} = n^+ \), and for each \( t \in S \) choose \( a_t \in \mathfrak{A}_t \) with \( a_t \neq 0, a_t \neq 1 \). By Corollary 1.3

\[
\prod_{t \in S} g_t^*(a_t) = 0.
\]

However \( 0 \neq \prod_{t \in S} i_t(a_t) = \prod_{t \in S} h_n \circ g_t^*(a_t) \) so that \( h_n \) is not \( m \)-complete.

There is an interesting contrast between \( E_n \) and \( P_n \) (under the hypotheses of Corollary 1.6). Let \( \{ \{ i_t \}_{t \in T}, \mathcal{B} \} \) and \( \{ \{ j_t \}_{t \in T}, \mathcal{C} \} \) be elements of \( P_n \) with \( \{ \{ i_t \}_{t \in T}, \mathcal{B} \} \subseteq \{ \{ j_t \}_{t \in T}, \mathcal{C} \} \). It is known (see [2], p. 179) that if \( \{ \{ i_t \}_{t \in T}, \mathcal{B} \} \in P_n \), then \( \{ \{ j_t \}_{t \in T}, \mathcal{C} \} \in P_{n'} \). On the other hand if \( \{ \{ j_t \}_{t \in T}, \mathcal{C} \} \in E_n \) then we have \( \{ \{ i_t \}_{t \in T}, \mathcal{B} \} \in E_n \).

**Corollary 1.7.** Assume \( \tilde{T} > n \) and \( m > n \). Then \( E_n \cup P_{n'} \neq P_n \).

**Proof.** Let \( S \subseteq T \) with \( \tilde{S} = n^+ \). Choose, for each \( t \in S \), \( d_t \in \mathfrak{A}_t \) with \( d_t \neq 0, d_t \neq 1 \). Let \( d = \bigcap_{t \in S} g_t^*(d_t) \). Let \( \mathcal{F} \) be the field of subsets of \( X \) which is generated by \( \mathcal{F}_n^* \cup \{d\} \). Note that \( g_t^* \) is a complete isomorphism from \( \mathfrak{A}_t \) into \( \mathcal{F} \). Let \( \{ f, \mathcal{C} \} \) be any \( m \)-extension of \( \mathcal{F} \). It is easily seen that \( \{ \{ f \circ g_t^* \}_{t \in T}, \mathcal{C} \} \in P_n' \).

Consider the isomorphism \( h_n \) from \( \mathcal{F}_n^* \) into \( \mathcal{C} \). \( h_n \circ g_t^* = f \circ g_t^* \) for every \( t \in T \). By Corollary 1.3 \( \prod_{t \in S} g_t^*(d_t) = 0 \). However \( \prod_{t \in S} h_n \circ g_t^*(d_t) = f(d) \neq 0 \). Thus \( h_n \) is not \( m \)-complete and \( \{ \{ f \circ g_t^* \}_{t \in T}, \mathcal{C} \} \in E_n \).

In order to show that \( \{ \{ f \circ g_t^* \}_{t \in T}, \mathcal{C} \} \in P_{n'} \) it suffices to show that \( \prod_{t \in S} f \circ g_t^*(-d_t) = 0 \). In particular suppose \( b = \prod_{t \in S} g_t^*(-d_t) \neq 0 \). Since \( b \cdot d = 0 \) the definition of \( \mathcal{F} \) enables us to write \( b = \bigcup_{t \in S} b_t \cdot g_t^*(-d_t) \) with \( b_t \in \mathcal{F}_n^* \). Choose \( t_0 \in S \) such that \( 0 \neq b_t \cdot g_t^*(-d_t) \leq b \).

By Lemma 1.2 there is a nonzero element \( a = \bigcap_{t \in S} g_t^*(a_t) \) of \( G_n \) such that \( a \subseteq b_t \cdot g_t^*(-d_t) \). Now \( \tilde{S} \leq n \) and \( S = n^+ \) and it follows that \( a \not\subseteq b \). Thus \( \prod_{t \in S} g_t^*(-d_t) = 0 \) and since \( f \) is \( m \)-complete, \( \prod_{t \in S} f \circ g_t^*(-d_t) = 0 \).

We now consider the case \( n = m \). It is known that \( E_m = P_m \) if \( m = \aleph_0 \) (see [2], p. 190, Example D). In this example \( T \) is the two element set \( \{1, 2\} \), \( \mathfrak{A}_t \) and \( \mathfrak{A}_t \) are \( \sigma \)-complete Boolean algebras which satisfy the \( \sigma \)-chain condition. The Boolean \( \sigma \)-product \( \{ i_t, i_t, \mathcal{B} \} \) is such that the subalgebra \( \mathcal{B} \) of \( \mathcal{B} \) which is generated by \( i_t(\mathfrak{A}_t) \cup i_t(\mathfrak{A}_t) \)
is not a σ-regular subalgebra of $\mathcal{B}$. Let $\{f, \mathcal{C}\}$ be any $m$-extension of $\mathcal{B}$. It follows, using the σ-chain condition on $\mathcal{I}$, and $\mathcal{I}_2$, that $\{(f \circ i_1, f \circ i_2, \mathcal{C})\} \in P_m$. Since $T$ is finite $\{(g^*_1, g^*_2), \mathcal{F}_m^*\}^*$ is the Boolean product of $\{(\mathcal{I}_1, \mathcal{I}_2)\}$. Let $h$ be the homomorphism from $\mathcal{F}_m^*$ into $\mathcal{B}$ such that $h \circ g^*_1 = i_1$ and $h \circ g^*_2 = i_2$. Then $h$ is an isomorphism from $\mathcal{F}_m^*$ onto $\mathcal{B}$. Consider the isomorphism $h_m$ from $\mathcal{F}_m^*$ into $\mathcal{C}$, given by Theorem 1.4. $h_m = f \circ h$ since they agree on $g^*_1(\mathcal{I}_1) \cup g^*_2(\mathcal{I}_2)$. $h_m$ is not $m$-complete because $f(\mathcal{F}_m^*)$ is not $m$-regular in $\mathcal{C}$. Thus $\{(f \circ i_1, f \circ i_2, \mathcal{C})\} \in E_m$. We give a simple for the case $m \geq 2^{\aleph_0}$.

**Example 1.8.** Assume $m \geq 2^{\aleph_0}$ and let $T$ be a set of power $\aleph_0$. For each $t \in T$ let $\mathcal{I}_t$ be a Boolean algebra having exactly four elements. Let $\mathcal{E}_m$ be the free Boolean $m$-algebra on $\{\mathcal{I}_t: t \in T\}$. $\mathcal{D}$ is not $m$-representable (see [2], p. 134). For each $t \in T$ choose $d_t$ to be one of the atoms of $\mathcal{I}_t$. Let $i_t$ be the isomorphism from $\mathcal{I}_t$ into $\mathcal{B}$ such that $i_t(d_t) = d_t$. Then $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in P_m$. By Lemma 1.2 $\mathcal{F}_m^*$ is atomic, the atoms being all sets of the form $\bigcap_{t \in T} g^*_t(a_t)$, where for each $t \in T a_t$ is an atom of $\mathcal{I}_t$. Denote the set of atoms of $\mathcal{F}_m^*$ by $\{C_r: r \in R\}$, then $\overline{R} = 2^{\aleph_0}$. We consider the isomorphism $h_m$ from $\mathcal{F}_m^*$ into $\mathcal{B}$. For each $r \in R$, $h_m(c_r)$ is an atom of $\mathcal{B}$. To show this we define

$$\mathcal{A} = \{b \in \mathcal{B}: \text{for each } r \in R \text{ either } b \cdot h_m(c_r) = 0 \text{ or } h_m(c_r) \leq b\}.$$  

It is easily seen that $\mathcal{A}$ is an $m$-subalgebra of $\mathcal{B}$ which includes $\{D_t: t \in T\}$. Hence $\mathcal{A} = \mathcal{B}$. Finally, $h_m$ is not $m$-complete. For otherwise $\sum_{r \in R} h_m(c_r) = 1$, and $\mathcal{B}$ would be atomic and hence isomorphic to an $m$-field of sets.

2. We now consider the problem of the existence of a smallest element of $P$, relative to the quasi-ordering “$\leq$”. A minimal element of $P$ always exists and can be constructed as follows. Let $\{\{f_t\}_{t \in T}, \mathcal{C}\}$ be a Boolean product of $\{\mathcal{I}_t: t \in T\}$ and let $\{h, \mathcal{B}\}$ be an $m$-completion of $\mathcal{C}$. Then $\{\{h \circ f_t\}_{t \in T}, \mathcal{B}\}$ is a minimal element of $P$. We shall show that this product need not be a smallest element of $P$. Hence $P$ need not have a smallest element.

**Example 2.1.** Let $m$ be any infinite cardinal. Let $T = \aleph_0$ and suppose that for each $t \in T \mathcal{I}_t$ is a four element Boolean algebra. For each $t \in T$ choose $a_t$ to be one of the atoms of $\mathcal{I}_t$. $\mathcal{C}$ is a free Boolean algebra of power $\aleph_0$, one set of free generators being $\{f_t(a_t): t \in T\}$. $\mathcal{B}$ has a countable dense subset, in particular $\mathcal{B}$ satisfies the countable chain condition. Thus $\mathcal{B}$ is complete. It follows that $\mathcal{B}$ is isomorphic to the quotient algebra $\mathcal{F}/\mathcal{D}_0$ where $\mathcal{F}$ is the $\sigma$-field
of Borel subsets of the unit interval \( I = \{ x : 0 < x \leq 1 \} \) of real numbers and \( \Delta_0 \) is the ideal consisting of those Borel sets which are of the first category.

To show that \( \{ [g \circ f_t]_{t \in T}, \mathcal{B} \} \) is not a smallest element of \( P \) we construct another \((m-0)\) product as follows. Let \( G \) be the set of all halfopen intervals of the form \( \{ x : 0 < x \leq r \} \) such that \( r \) is rational and \( 0 < r \leq 1 \). \( \mathcal{F} \) is \( \sigma \)-generated by \( G \). The subalgebra \( \mathcal{F}_0 \) of \( \mathcal{F} \) which is generated by \( G \) is denumerable and atomless. Hence \( \mathcal{F}_0 \) is isomorphic to \( \mathbb{C} \) (see [1], p. 54). Let \( g \) be an isomorphism from \( \mathbb{C} \) onto \( \mathcal{F}_0 \). Let \( \Delta_1 \) be the ideal of \( \mathcal{F} \) consisting of those Borel sets having Lebesgue measure 0. We note that \( \mathcal{F}_0 \cap \Delta_1 = \{ 0 \} \). Finally for each \( t \in T \) let \( h_t \) be the isomorphism from \( \mathcal{A}_t \) into \( \mathcal{F} / \Delta_1 \) defined by \( h_t(a_t) = [g \circ f_t(a_t)] \Delta_1 \). It is easily seen that \( \{ [h_t]_{t \in T}, \mathcal{F} / \Delta_1 \} \in P \).

Now assume \( \{ [h \circ f_t]_{t \in T}, \mathcal{B} \} \leq \{ [h_t]_{t \in T}, \mathcal{F} / \Delta_1 \} \). Then there is an \( m \)-homomorphism \( p \) from \( \mathcal{F} / \Delta_1 \) onto \( \mathbb{C} / \Delta_0 \). Since \( \mathcal{F} / \Delta_1 \) satisfies the countable chain condition the kernel of \( p \) is a principal ideal. \( \mathcal{F} / \Delta_1 \) is isomorphic to a principal ideal of \( \mathbb{C} / \Delta_0 \). However \( \mathbb{C} / \Delta_0 \) is homogeneous (see [2], p. 105). Thus \( \mathcal{F} / \Delta_0 \) is isomorphic to \( \mathbb{C} / \Delta_1 \), which is a contradiction.

Next we consider the problem of the existence of a smallest element of \( P_n \). Let \( \{ g, \mathcal{B} \} \) be an \( m \)-completion of \( \mathcal{F}^*_n \). Then \( \{ [g \circ g_t^*]_{t \in T}, \mathcal{B} \} \) is a minimal element of \( P_n \). Also it is known (see [2], p. 183) that if all the \( \mathcal{A}_t \) are \( m \)-representable then there is an \((m-n)\) product \( \{ [i_t]_{t \in T}, \mathbb{C} \} \) for which \( \mathbb{C} \) is \( m \)-representable. We give an example of \( \{ \mathcal{A}_t \}_{t \in T} \) for which \( \mathcal{B} \) is not \( m \)-representable and \( \{ [g \circ g_t^*]_{t \in T}, \mathcal{B} \} \) is not a smallest element of \( P_n \).

**Example 2.2.** Assume that \( m \geq 2^{n+1} \). Let \( T = n^+ \) and for each \( t \in T \) let \( \mathcal{A}_t \) be a four element Boolean algebra. We show that \( \mathcal{B} \) is not \( n^+ \)-distributive. Choose, for each \( t \in T, a_t \) to be one of the atoms of \( \mathcal{A}_t \). Then

\[
\prod_{t \in T} (g \circ g_t^*(a_t) + - g \circ g_t^*(a_t)) = 1 .
\]

However for each function \( \eta \in H^T \) (here \( H = \{ +1, -1 \} \)) we have

\[
\prod_{t \in T} \eta(t) \cdot g_t^*(a_t) = 0 .
\]

This follows from Corollary 1.3. Thus \( \prod_{t \in T} \eta(t) \cdot g \circ g_t^*(a_t) = 0 \). This proves \( \mathcal{B} \) is not \( n^+ \)-distributive and hence not \( m \)-representable.

To show that \( \{ [g \circ g_t^*]_{t \in T}, \mathcal{B} \} \) is not a smallest element of \( P_n \), let \( \{ [i_t]_{t \in T}, \mathbb{C} \} \) be any \((m-n)\) product of \( \{ \mathcal{A}_t \}_{t \in T} \) such that \( \mathbb{C} \) is \( m \)-representable. \( \mathcal{B} \) is not an \( m \)-homomorphic image of \( \mathbb{C} \). Thus the inequality
\{\{g \circ g_i^*\}_{i \in T}, \mathcal{B}\} \subseteq \{\{i\}_{i \in T}, \mathcal{C}\}

does not hold.

**References**


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