PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

# THE BENDING OF SPACE CURVES INTO PIECEWISE HELICAL CURVES

JAMES M. SLOSS

**It is the purpose of this paper to show that a regular $C^3$ space curve $\Gamma$ in a Euclidean 3-space, whose curvature $\kappa \neq 0$, can be bent into a piecewise helix (i.e., a curve that is a helix but for a finite number of corners) in such a way that the piecewise helix remains within a tubular region about $C$ of arbitrarily small preassigned radius. Moreover, we shall show that the bending can be carried out in such a way that either (a) the piecewise helix is circular or (b) the piecewise helix has the same curvature as $\Gamma$ at corresponding points except possibly at corners, of (c) if the torsion of $\Gamma$ is nowhere zero, then the piecewise helix has the same torsion as $\Gamma$ at corresponding points except possibly at corners.**

Also we shall show that if, in addition, $\Gamma$ has a bounded fourth derivative, then an explicit formula can be given for a sufficient number $n$ of helices that make up the piecewise helix, where $n$ depends on $\Gamma$ and the radius of the tubular region about $\Gamma$. In this case, we shall also show how the determination of the piecewise helix can be reduced to a problem in simple integration.

## 1. Bendability.

DEFINITION 1. A curve is called a *piecewise helix* if it consists of a finite number of segments, each of which is a helix (i.e., a curve whose tangent makes a constant angle with a fixed direction). A point at which two consecutive helices meet will be called a *corner* of the piecewise helix.

REMARK. If, in particular, between corners the helix is a circular helix, then the piecewise helix will be called a *piecewise circular helix*.

THEOREM 1. *Let $\Gamma$: $r(s)$, $s =$ arc length, $0 \leqq s \leqq l$, be a regular $C^3[0, l]$*[1] *curve whose curvature $\kappa(s)$ is nowhere zero. Then for any given $\varepsilon > 0$*

(a) *there exists a piecewise circular helix $\Gamma_1^*$: $h_1^*(s)$, $s =$ arc length, $0 \leqq s \leqq l$, such that:*

$$|r(s) - h_1^*(s)| < \varepsilon\,, \qquad 0 \leqq s \leqq l\,;$$

[1] (I.e., $r(s)$ can be extended to lie in $C^3$ on some open set containing $0 \leqq s \leqq l$.)

(b) *there exists a piecewise helix* $\Gamma_2^*$: $h_2^*(s)$, $s =$ *arc length*, $0 \leqq s \leqq l$, *such that*:

$$|r(s) - h_2^*(s)| < \varepsilon\,, \qquad 0 \leqq s \leqq l$$

*and* $\Gamma_2^*$ *has the same curvature as* $\Gamma$ *at corresponding points, except possibly at the corners of* $h_2^*(s)$;

(c) *provided the torsion* $\tau(s)$ *is nowhere zero, there exists a piecewise helix* $\Gamma_3^*$: $h_3^*(s)$, $s =$ *arc length*, $0 \leqq s \leqq l$, *such that*:

$$|r(s) - h_3^*(s)| < \varepsilon\,, \qquad 0 \leqq s \leqq l$$

*and* $\Gamma_3^*$ *has the same torsion as* $\Gamma$ *at corresponding points, except possibly at corners of* $h_3^*(s)$.

REMARK. In each case the curve $\Gamma$ is "bent" into a piecewise helix.

*Proof*. We shall prove (b) and indicate what minor modifications are necessary to prove (a) and (c). Let $\kappa(s)$ and $\tau(s)$ be the curvature and torsion respectively of $\Gamma$. Then $\kappa(s) \in C^1[0, l]$ and $\tau(s) \in C^0[0, l]$ since $r(s) \in C^3[0, l]$. By hypothesis, $\kappa(s) \neq 0$; therefore,

$$f(s) = \frac{\tau(s)}{\kappa(s)}$$

is continuous and thus uniformly continuous on $[0, l]$. Let

$$|\kappa(s)| \leqq \kappa_{\max} \qquad \text{on } 0 \leqq s \leqq l \tag{1.1}$$

and

$$|f(s)| \leqq f_{\max} \qquad \text{on } 0 \leqq s \leqq l \tag{1.2}$$

and choose $\delta(\varepsilon) > 0$ such that

$$|f(s_2) - f(s_1)| < \alpha\varepsilon \tag{1.3}$$

provided $|s_2 - s_1| \leqq \delta$, where

$$\alpha = \langle \kappa_{\max} l^2 \sqrt{6} \exp\{l\kappa_{\max}\sqrt{2(1 + f_{\max}^2)}\}\rangle^{-1}\,. \tag{1.4}$$

Let

$$n = n(\varepsilon) = \text{smallest integer} \geqq \frac{l}{\delta}$$

and

$$\begin{aligned} I_0 &= \{s\colon 0 \leqq s \leqq \delta\} \\ I_j &= \{s\colon j\delta < s \leqq (j+1)\delta\}\,, \qquad j = 1, 2, \cdots, n-2 \\ I_{n-1} &= \{s\colon (n-1)\delta < s \leqq l\}\,. \end{aligned} \tag{1.4.1}$$

Then $I_j (0 \leqq j \leqq n-1)$ form a disjoint covering of $[0, l]$, each of length $\leqq \delta$.

Let

$$\tau_j(s) = f_j \kappa(s),\ s \in I_j\ , \qquad j = 0, 1, \cdots, n-1 \tag{1.5}$$

where

$$f_j = \begin{cases} f[(j+1)\delta] & \text{for } j = 0, 1, \cdots, n-2\ . \\ f[l] & \text{for } j = n-1 \end{cases}$$

By the fundamental theorem for space curves there exists a unique curve $h_j(s)$, $s \in I_j$, for which:

(i) its curvature and torsion are respectively $\kappa(s)$ and $\tau_j(s)$ as defined by (1.5), and

(ii) its position $h_j(s)$, tangent $t_j(s)$, principal normal $n_j(s)$ and binomial $b_j(s)$ satisfy the initial conditions:

$$h_j(j\delta) = r(j\delta),\ t_j(j\delta) = e_1(j\delta),\ n_j(j\delta) = e_2(j\delta),\ b_j(j\delta) = e_3(j\delta) \tag{1.6}$$

where $e_1(s)$, $e_2(s)$ and $e_3(s)$ are the tangent, principal normal and binormal of $r(s)$ respectively, and $s$ is the arc length parameter of $h_j$.

Moreover, if

$$\Phi_j(s) = \begin{bmatrix} t_j(s) \\ n_j(s) \\ b_j(s) \end{bmatrix}, \qquad A_j = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & f_j \\ 0 & -f_j & 0 \end{bmatrix}, \tag{1.7}$$

then $\Phi_j(s)$ satisfies the differential equation:

$$\Phi_j'(s) = \kappa(s) A_j \Phi_j(s)\ . \tag{1.8}$$

Also, because $\tau_j(s)/\kappa(s) = f_j =$ constant on $I_j$, $h_j(s)$ is a helix on $I_j$.

By the Frenet formulae for $\Gamma$, we have:

$$\Psi'(s) = \kappa(s) A(s) \Psi(s)\ , \qquad 0 \leqq s \leqq l\ , \tag{1.9}$$

where

$$\Psi(s) = \begin{bmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{bmatrix},\ A(s) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & f(s) \\ 0 & -f(s) & 0 \end{bmatrix}. \tag{1.10}$$

Considering both (1.8) and (1.9) as differential equations on $I_j$, we obtain:

$$\Phi_j(s) = \Phi_j(j\delta) + \int_{j\delta}^{s} \kappa(t) A_j \Phi_j(t) dt,\ s \in I_j\ , \qquad j = 0, 1, \cdots, n-1 \tag{1.11}$$

and

$$\Psi(s) = \Psi(j\delta) + \int_{j\delta}^{s} \kappa(t)A(t)\Psi(t)dt,\ s \in I_j\ , \qquad j = 0, 1, \cdots, n-1 \tag{1.12}$$

Since by (1.6) $\Phi_j(j\delta) = \Psi(j\delta)$, we see that if

$$||(c_{ij})|| = \sqrt{\sum_{i,j=1}^{3} c_{ij}^2}\ ,$$

then

$$\begin{aligned} ||\Psi(s) - \Phi_j(s)|| \leqq \int_{j\delta}^{s} |\kappa(t)|\, ||A(t) - A_j||\, ||\Psi(t)||\, dt \\ + \int_{j\delta}^{s} |\kappa(t)|\, ||A_j||\, ||\Psi(t) - \Phi_j(t)||\, dt\ . \end{aligned} \tag{1.13}$$

But by (1.7), (1.10), and (1.3)

$$||A(t) - A_j|| = \sqrt{2[f(t) - f_j]^2} < \sqrt{2}\alpha\varepsilon \qquad \text{for } t \in I_j\ .$$

Also we have

$$||\Psi(t)|| = \sqrt{3}$$

and by (1.7)

$$||A_j|| = \sqrt{2(1 + f_j^2)} \leqq \sqrt{2(1 + f_{\max}^2)}\ .$$

Thus

$$||\Psi(s) - \Phi_j(s)|| < M\delta\varepsilon\alpha + N\int_{j\delta}^{s} ||\Psi(t) - \Phi_j(t)||\, dt\ , \qquad s \in I_j\ , \tag{1.14}$$

where

$$\begin{aligned} M &= \kappa_{\max}\sqrt{6} \\ N &= \kappa_{\max}\sqrt{2(1 + f_{\max}^2)}\ . \end{aligned}$$

Let

$$C = \sup_{t \in I_j} ||\Psi(t) - \Phi_j(t)||\ ,$$

then by (1.14)

$$||\Psi(s) - \Phi_j(s)|| < M\delta\varepsilon\alpha + NC(s - j\delta) \tag{1.15}$$

from which we see upon combining (1.14) and (1.15) that

$$\begin{aligned}
&\|\Psi(s) - \Phi_j(s)\| \\
&\quad < M\delta\varepsilon\alpha\left[1 + N(s - j\delta) + N^2\frac{(s - j\delta)^2}{2!} + \cdots + N^k\frac{(s - j\delta)^k}{k!}\right] \\
(1.16) &\qquad\qquad + \frac{(s - j\delta)^{k+1}}{(k + 1)!} CN^{k+1} < M\delta\varepsilon\alpha e^{N\delta} \\
&\qquad\qquad < Ml\varepsilon\alpha e^{Nl} \\
&\qquad\qquad < \varepsilon/l\ , \qquad\qquad s \in I_j\ ,
\end{aligned}$$

by the definition (1.4) of $\alpha$.

If we let

$$\Phi^*(s) \equiv \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix} = \Phi_j(s) \equiv \begin{bmatrix} t_j(s) \\ n_j(s) \\ b_j(s) \end{bmatrix},\ s \in I_j\ , \qquad j = 0, 1, \cdots, n = 1\ ,$$

then $\Phi^*(s)$ is piecewise continuous on $[0, l]$ with discontinuities possibly at

$$s = j\delta\ , \qquad j = 0, 1, 2\cdots, n - 1\ ,$$

and by (1.16) since $I_j$ is a cover of $[0, l]$,

$$\|\Psi(s) - \Phi^*(s)\| < \varepsilon/l \qquad \text{for } 0 \leqq s \leqq l\ . \tag{1.17}$$

Let

$$h^*(s) = r(0) + \int_0^s t(\sigma)d\sigma\ , \qquad \text{for } 0 \leqq s \leqq l\ . \tag{1.18}$$

Then $h^*(s)$ is a piecewise helix $\Gamma_2^*$ for which

$$h^{*\prime}(s) = h_j'(s)\ , \qquad \text{for } s \in I_j, j = 0, 1, \cdots, n - 1\ .$$

Thus for $0 \leqq s \leqq l$:

$$\begin{aligned}
|r(s) - h^*(s)| &\leqq \int_0^s |e_1(s) - t(s)|\, ds \\
&\leqq \int_0^l \|\Psi(s) - \Phi^*(s)\|\, ds \\
&< \varepsilon
\end{aligned}$$

by (1.17).

Next we note that $s$ is the arc length of $h^*(s)$ since

$$|h^{*\prime}(s)| = |h_j'(s)| = |t_j(s)| = 1\ , \qquad \text{for } s \in I_j\ .$$

Moreover on the interior of $I_j$:

$$|h^{*\prime\prime}(s)| = |h_j''(s)| = \text{curvature} = \kappa(s)$$

by construction of $h_j(s)$.

This completes the proof of part (b). For the proof of part (a) and part (c), only obvious slight modifications are necessary. In part (a), we need only the additional fact that a helix is circular if the curvature and torsion are both constant.

## 2. Explicit results.

If we allow $r(s)$ to have one more bounded derivative we have:

THEOREM 2. *If in addition to the assumptions of Theorem* 1, *we also assume that* $r(s)$ *has a bounded fourth derivative on* $[0, l]$, *then we can choose* $n(\varepsilon)$ *in part* 2 *to be*

$$n(\varepsilon) = \textit{smallest integer} > \frac{g^* l}{\alpha\varepsilon} \tag{2.1}$$

*where*

$$\alpha = \langle \kappa_{\max} l^2 \sqrt{6} \exp\{l\kappa_{\max}\sqrt{2(1+f_{\max}^2)}\}\rangle^{-1} \tag{2.2}$$

$$\begin{aligned} &\left|\frac{r'\cdot(r''\times r'''')}{[r''\cdot r'']^{3/2}} - \frac{3[r'''\cdot r''][r'\cdot(r''\times r''')]}{[r''\cdot r'']^{5/2}}\right| < g^*, \quad 0 \leqq s \leqq l\,, \\ &\left|\frac{r'\cdot(r''\times r''')}{[r''\cdot r'']^{3/2}}\right| < f_{\max},\ [r''\cdot r'']^{1/2} < \kappa_{\max}\,, \qquad 0 \leqq s \leqq l\,, \end{aligned} \tag{2.3}$$

REMARK. A similar result holds for parts (a) and (c).

*Proof.* Since

$$\kappa(s) = [r''(s)\cdot r''(s)]^{1/2}$$

and

$$\tau(s) = \frac{r'\cdot(r''\times r''')}{r''\cdot r''}$$

the expression in the first inequality in (2.3) is simply the derivative of

$$f(s) = \frac{\tau(s)}{\kappa(s)}\,.$$

Thus

$$|f(s_2) - f(s_1)| = \left|\int_{s_1}^{s_2} f'(s)ds\right| < g^*[s_2 - s_1]\,.$$

If we choose

$$\delta = \frac{\alpha\varepsilon}{g^*}\,, \tag{2.4}$$

where $\alpha$ is given by (2.2) = (1.4), then

$$|f(s_2) - f(s_1)| < \alpha\varepsilon$$

whenever $|s_2 - s_1| < \delta$. This, by the proof of part (b) of Theorem 1, gives the result since

$$n(\varepsilon) = \text{smallest integer} > \frac{g^* l}{\alpha\varepsilon} = \frac{l}{\delta}\,.$$

THEOREM 3. *Let $\Gamma$: $r(s)$, $s$ = arc length, $0 \leqq s \leqq l$, be a regular space curve with bounded fourth derivative and nowhere-zero curvature. Denote the curvature, torsion, tangent, principal normal and binormal of $\Gamma$ by $\kappa(s)$, $\tau(s)$, $e_1(s)$, $e_2(s)$ and $e_3(s)$. For any given $\varepsilon > 0$, let $n(\varepsilon)$, $\delta$, and $I_j$ ($j = 0, 1, \cdots, n$) be given by* (2.1), (2.4) *and* (1.4.1), *respectively. Put*

$$t_j(s) = \frac{1}{m^2}\{[f_j^2 + \cos(g_j(s)m)]e_1(j\delta) + [m\sin(g_j(s)m)]e_2(j\delta) + f_j[1 - \cos(g_j(s)m)]e_3(j\delta)\}$$

*where*

$$f_j = \tau[(j+1)\delta]\}/\{\kappa[(j+1)\delta]\},\ m = +\sqrt{1+f_j^2},\ g_j(s) = \int_{j\delta}^{s}\kappa(\sigma)d\sigma$$

*and let*

$$t(s) = t_j(s),\ s \in I_j,\ j = 0, 1, \cdots, n\,.$$

*Then the curve*

$$\Gamma^*\colon h^*(s) = r(0) + \int_0^s t(\sigma)d\sigma,\ s = \text{arc length},\ 0 \leqq s \leqq l\,,$$

*is a piecewise helix such that*

$$|r(s) - h^*(s)| < \varepsilon,\ 0 \leqq s \leqq l\,,$$

*and $\Gamma^*$ has the same curvature as $\Gamma$ at corresponding points except possibly at the corners.*

*Proof.* From (1.7)

$$\Phi_j(s) = \begin{bmatrix} t_j(s) \\ n_j(s) \\ b_j(s) \end{bmatrix}$$

satisfies the system of differential equations

$$\Phi_j'(s) = \kappa(s) A_j \Phi_j(s) \qquad \text{on } I_j \,, \tag{1.8}$$

where $A_j$ is given by (1.7). The solution of (1.8) for which $\Phi_j(j\delta) = \Psi(j\delta)$ is given by

$$\Phi_j(s) = e^{g_j(s) A_j} \Psi(j\delta) \,.$$

The eigenvalues of $A_j$ are 0, $im$ and $-im$ and the corresponding eigenvectors are:

$$T_1 = \begin{bmatrix} f_j \\ 0 \\ 1 \end{bmatrix}, \; T_2 = \begin{bmatrix} 1 \\ im \\ -f_j \end{bmatrix}, \; T_3 = \begin{bmatrix} 1 \\ -im \\ -f_j \end{bmatrix} .$$

Also the matrix $T = (T_1, T_2, T_3)$ has the inverse

$$T^{-1} = \frac{1}{2m^2} \begin{bmatrix} 2f_j & 0 & 2 \\ 1 & -im & -f_j \\ 1 & im & -f_j \end{bmatrix} .$$

Thus

$$T^{-1} e^{g_j(s) A_j} T = e^{g_j(s) D_j} \,,$$

where

$$D_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & im & 0 \\ 0 & 0 & -im \end{bmatrix}$$

and

$$\begin{aligned} e^{g_j(s) A_j} &= T e^{g_j(s) D_j} T^{-1} \\ &= \frac{1}{m^2} \begin{bmatrix} f_j^2 + \cos g_j(s)m, & m \sin g_j(s)m, & f_j(1 - \cos g_j(s)m) \\ * & * & * \\ * & * & * \end{bmatrix} . \end{aligned}$$

From this it follows that

$$\begin{bmatrix} t_j(s) \\ n_j(s) \\ b_j(s) \end{bmatrix} \equiv \Phi_j(s) = e^{g_j(s) A_j} \begin{bmatrix} e_1(j\delta) \\ e_2(j\delta) \\ e_3(j\delta) \end{bmatrix} = \frac{1}{m^2} \times$$

$$\times \begin{bmatrix} [f_j^2 + \cos g_j(s)m] e_1(j\delta) + [m \sin g_j(s)m] e_2(j\delta) + f_j[1 - \cos g_j(s)m] e_3(j\delta) \\ * \quad\quad * \quad\quad * \\ * \quad\quad * \quad\quad * \end{bmatrix}$$

which gives (2.5) and the theorem is proved.

REMARKS. By using the definition of torsion as given by Hartman and Wintner [1], p. 771, [3] p. 202, the continuity requirement of Theorem 1 can be relaxed from $C^3$ to $C^2$. A question of further interest would be to consider the bending of normal curves, see for example, Nomizu [2] and Wong and Lai [4].

## BIBLIOGRAPHY

1. P. Hartman and A. Wintner, *On the fundamental equations of differential geometry* Amer. J. Math. **72** (1950), 757-774.
2. K. Nomizu, *On Frenet equations for curves of class $C^\infty$*, Tohoku J. Math. **11** (1959), 106-112.
3. A. Wintner, *On the infinitesimal geometry of curves*, Amer. J. Math. **75** (1953).
4. Y. C. Wong and H. F. Lai, *A critical examination of the theory of curves in three dimensional differential geometry*, Tohoku J. Math. **19** (1967), 1-31.

Received March 17, 1969, and revised form June 13, 1969. The author wishes to acknowledge support for this research by the National Aeronautics Space Administration, NASA Grant NGR 05-010-008.

UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA