SOME REMARKS ON CLIFFORD'S THEOREM
AND THE SCHUR INDEX

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Some time ago Clifford described the behavior of an irreducible representation of a finite group when it is restricted to a normal subgroup. One interesting case in this description requires that the representation be written in an algebraically closed field. In this note we shall consider this case when the field is "small". We describe conditions under which an irreducible representation decomposes as the tensor product of two projective representations. Our approach uses certain subalgebras of the group algebra and the course of the discussion makes it fairly easy to keep track of the division algebras that appear. Hence we obtain some information about the Schur index. We apply this information to the case where the group is a semi-direct product PA of a p-group P and a normal cyclic group A. If $\mathscr{F}$ is an algebraic number field and χ an absolutely irreducible character of PA, then there normal subgroups $P_1 \supseteq P_2 \supseteq P_3$ of P which contain $C_P(A)$ such that the Schur index $m_{\mathscr{F}}(\chi)$ of χ over $\mathscr{F}$ divides $2[P_1 : P_2]e$ where e is the exponent of P_2/P_3. The factor 2 can be omitted if $p \neq 2$. Some conditions are available to restrict the P_i further.

1. **Preliminaries.** In this section we summarize the results about the Schur index and Clifford's theory that will be used later.

Let G be a finite group, $\mathscr{F}$ a field of characteristic zero, M an irreducible $\mathscr{F}(G)$-module with character θ.

(1.1) There are absolutely irreducible (complex-valued) characters $\chi_1, \cdots, \chi_k$ such that $\theta = m(\chi_1 + \cdots + \chi_k)$.

(1.2) Let $\mathscr{F}(\chi_i)$ denote the field generated over $\mathscr{F}$ by the values of χ_i on G. Then $\mathscr{F}(\chi_1)$ is a normal extension of $\mathscr{F}$ and for each $i = 1, \cdots, k$, there is a unique element of the Galois group of $\mathscr{F}(\chi_1)$ over $\mathscr{F}$ which carries χ_1 to χ_i. In particular $(\mathscr{F}(\chi_1) : \mathscr{F}) = k$.

(1.3) The integer m is called the *Schur index* of χ_1 over $\mathscr{F}$ and is denoted by $m_{\mathscr{F}}(\chi_1)$. The division algebra $D = \mathrm{End}_{\mathscr{F}(G)}(M)$ has center isomorphic to $\mathscr{F}(\chi_1)$ and the dimension of D over its center is m^2.

One remark on terminology. A matrix ring over D is said to have *index* m if D has dimension m^2 over its center.

The proofs of these statements are available in several places; see for example Curtis and Reiner [2] or Fein [5].

Now let H be a normal subgroup of G. Clifford's theory tells how M behaves as a module over $\mathscr{F}(H)$.

(1.4) $M_H \cong n(V_1 \oplus \cdots \oplus V_s)$ where the V_i are mutually non-isomorphic irreducible $\mathscr{F}(H)$-modules, conjugate under the action of G. Here the coefficient n means direct sum of n copies of $V_1 \oplus \cdots \oplus V_s$.

(1.5) Let $I_1 = \{x \in G \mid xV_1 \cong V_1 \text{ as } \mathscr{F}(H)\text{-modules}\}$. Then there exists an irredubible $\mathscr{F}(I_1)$-module W_1 such that $(W_1)_H \cong nV_1$ and the induced module $W_1^G \cong M$.

In the case where $s > 1$, I_1 is a proper subgroup of G and the original module is induced from the $\mathscr{F}(I_1)$-module. Hence some questions can be answered by induction. In case $s = 1$ there is no induction but in its place we have the following.

(1.6) Suppose $\mathscr{F}$ is algebraically closed and $s = 1$ in (1.4). Then the representation afforded by M decomposes into the tensor product of two (irreducible) projective representations of G one of which can be viewed as a projective representation of G/H. The one representation has dimension the same as the dimension of V_1, the other, has dimension n.

2. **Clifford's Theorem in the general case.** We shall continue to use the notation introduced in §1. However we assume $M_H \cong nV$ with $V = V_1$ in (1.4). We shall made one assumption that will simplify the following discussion considerably. Namely we assume that $\mathscr{F}$ contains the values of the character χ_1. Then in view of (1.2) we have $\theta = m\chi$ where $\chi = \chi_1$ in (1.1).

Let V have character γ and suppose

(2.1) $\gamma = m_H(\varphi_1 + \cdots + \varphi_t)$

is the decomposition of γ into absolutely irreducible characters of H.

In the group algebra $\mathscr{F}(G)$ let $e(\theta)$ denote the central idempotent which acts as the identity on M and such that $\mathscr{F}(G)e(\theta)$ is a simple algebra; let $e(\gamma)$ denote the centrally primitive idempotent in $\mathscr{F}(H)$ corresponding to V. The condition that $M_H \cong nV$ implies $\gamma(h) = \gamma(x^{-1}hx)$ for all h in H and x in G. It follows that $e(\gamma)$ belongs to the center of $\mathscr{F}(G)$. Also $e(\theta)e(\gamma) \neq 0$ because both act as the identity on V so $e(\theta)e(\gamma)$ is a nonzero central idempotent in $\mathscr{F}(G)e(\theta)$. By simplicity we must have $e(\theta)e(\gamma) = e(\theta)$. Thus multiplication by $e(\theta)$ sends the simple algebra $\mathscr{F}(H)e(\gamma)$ onto the nonzero subalgebra

$\mathscr{F}(H)e(\theta)$ of $\mathscr{F}(G)e(\theta)$ both having the same identity, $e(\theta)$. We note that by (1.3) and (2.1) the center $\mathscr{L}$ of $\mathscr{F}(H)e(\theta)$ is isomorphic to $\mathscr{F}(\varphi_1)$. So we have proved

LEMMA 2.2. $\mathscr{F}(H)e(\theta)$ *is a simple algebra with center* $\mathscr{L}$ *isomorphic to* $\mathscr{F}(\varphi_1)$.

Each element of G acts by conjugation on $\mathscr{F}(H)e(\theta)$ since $e(\theta)$ is central and H is a normal subgroup. Thus G also acts on the center $\mathscr{L}$ of $\mathscr{F}(H)e(\theta)$ as a group fixing $\mathscr{F}$. Let I denote the kernel of the action of G on $\mathscr{L}$ so that I is normal in G and G/I is a group of $\mathscr{F}$-automorphisms of $\mathscr{L}$. Let $\{y_i\}1 \leq i \leq r$ be a set of representatives of the cosets of I in G and let y_i induce the automorphism σ_i oh $\mathscr{L}$.

LEMMA 2.3. *The elements* $y_i e(\theta)$ *are independent over* $\mathscr{F}(I)e(\theta)$ *and* $\mathscr{F}(G)e(\theta) = \sum \mathscr{F}(I)e(\theta)y_i$.

Proof. Suppose there exist elements $\{\alpha_j\}$ in $\mathscr{F}(G)e(\theta)$ which centralize $\mathscr{L}$ and with the properties

(a) $\sum\limits_{j=1}^{s} \alpha_j y_j e(\theta) = 0$.

(b) $\alpha_j \neq 0$ for each j.

(c) The integer s is minimal with respect to (a) and (b).

Then for any z in $\mathscr{L}$ we have $\sum \alpha_j z y_j = \sum \alpha_j y_j z = 0$. Then y_j induces σ_j $\mathscr{L}$ so

(2.4) $$\sum_{j=1}^{s} \alpha_j(z - \sigma_j(z))y_j = 0 \ .$$

There is no loss of generality in assuming that $y_1 = 1$ since a change of coset representatives can always bring this about. Thus $z - \sigma_1(z) = 0$ and the relation (2.4) has fewer than s nonzero terms. By the choice of s it follows $\alpha_j(z - \sigma_j(z)) = 0$ for each j and all z in $\mathscr{L}$. If $j > 1$ then $\sigma_j \neq$ identity so there exists z in $\mathscr{L}$ with $\sigma_j(z) \neq z$. But then $\alpha_j = 0$ contrary to (b). Hence $s = 1$ but this is also contrary to (b). Thus no such relation exists. Since $\mathscr{F}(I)e(\theta)$ centralizes $\mathscr{L}$ we have proved the independence of the $y_i e(\theta)$ over $\mathscr{F}(I)e(\theta)$. The second part of the lemma is clear.

COROLLARY 2.5. $\mathscr{F}(I)e(\theta)$ *is the full centralizer of* $\mathscr{L}$ *in* $\mathscr{F}(G)e(\theta)$ *so* $\mathscr{F}(I)e(\theta)$ *is a simple algebra with center* $\mathscr{L}$ *and dimension* $(\chi(1)/t)^2$ *over* $\mathscr{L}$.

Proof. The lemma shows the $y_i e(\theta)$ are independent over the

centralizer of $\mathscr{L}$ and so the proper inclusion of $\mathscr{F}(I)e(\theta)$ into the centralizer of $\mathscr{L}$ would make the equality $\mathscr{F}(G)e(\theta) = \sum \mathscr{F}(I)e(\theta)y_i$ impossible. The remaining statements follow from Albert, Theorem 12, page 53 [1] and the facts that $(\mathscr{F}(G)e(\theta):\mathscr{F}) = \chi(1)^2$ and $(\mathscr{L}:\mathscr{F}) = t$ (see (1.2) and (2.2)).

COROLLARY 2.6. $[G:I] = t$ so $G/I \cong$ Galois group $(\mathscr{L}/\mathscr{F})$.

Proof. The result follows at once if we use (2.3) and (2.5) to compute the dimension of $\mathscr{F}(G)e(\theta)$ over $\mathscr{F}$ along with the fact that this dimension is $\chi(1)^2$.

Now let $\chi \mid H = a(\varphi_1 + \cdots + \varphi_t)$. Since $\theta = m\chi$ and $\theta \mid H = n\gamma$, equation (2.1) implies $a = nm_H/m$. Let I_1 denote the inertial group of φ_1; that is

$$I_1 = \{x \in G \mid \varphi_1(h) = \varphi_1(x^{-1}hx) \quad \text{for all} \quad h \in H\} \ .$$

COROLLARY 2.7. $I_1 = I$.

Proof. The irreducible characters of H appearing in $\chi \mid H$ are conjugate under the action of G so that $[G:I_1] = t = [G:I]$. Thus it is sufficient to show $I_1 \subseteq I$. Let $\mathscr{F}_1 = \mathscr{F}(\varphi_1)$ and notice that in $\mathscr{F}_1(H)$ we have $e(\gamma) = e(\varphi_1) + \cdots + e(\varphi_t)$ where $e(\varphi_i)$ is the central idempotent of $\mathscr{F}_1(H)$ corresponding to φ_i. Recall from above that $e(\gamma)e(\theta) = e(\theta)$ so

$$\mathscr{F}(H)e(\theta) = \mathscr{F}(H)(e(\varphi_1) + \cdots + e(\varphi_t))e(\theta) \xrightarrow{\ R\ } \mathscr{F}_1(H)e(\varphi_1)e(\theta)$$

where R denotes right multiplication by $e(\varphi_1)$. The map R is a ring isomorphism. One point requires further comment. The obvious range of R is $\mathscr{F}(H)e(\varphi_1)e(\theta)$ rather than $\mathscr{F}_1(H)e(\varphi_1)e(\theta)$. However we can prove these are equal in the following way. Certainly $\mathscr{F} \subseteq \mathscr{F}_1$ so $\mathscr{F}(H)e(\varphi_1)e(\theta) \subseteq \mathscr{F}_1(H)e(\varphi_1)e(\theta)$. We prove equality by computing the $\mathscr{F}$-dimension of both sides. Since multiplication by $e(\theta)$ gives an $\mathscr{F}_1$-algebra isomorphism of $\mathscr{F}_1(H)e(\varphi_1)$ onto $\mathscr{F}_1(H)e(\varphi_1)e(\theta)$ we see the latter algebera has $\mathscr{F}_1$-dimension equal to $\varphi_1(1)^2$ and $\mathscr{F}$-dimension $t\varphi_1(1)^2$. To compute $\mathscr{F}$-dimension of $\mathscr{F}(H)e(\varphi_1)e(\theta)$ we first note $\mathscr{F}$-dimension of $\mathscr{F}(H)e(\gamma)e(\theta)$ equals $\mathscr{F}_1$-dimension of $\mathscr{F}_1(H)e(\gamma)e(\theta)$ because the latter algebra is obtained by extending the scalar field from $\mathscr{F}$ to $\mathscr{F}_1$. Now the algebra $\mathscr{F}_1(H)e(\gamma)e(\theta)$ equals $\sum \mathscr{F}_1(H)e(\varphi_i)e(\theta)$ and this has $\mathscr{F}$-dimension $t\varphi_1(1)^2$ as we wanted.

For an element x in I_1 we have $x^{-1}e(\varphi_1)x = e(\varphi_1)$ so the map R commutes with the action of I_1 on the two algebras in question. It is clear that $\mathscr{F}_1$ is the center of $\mathscr{F}_1(H)e(\varphi_1)e(\theta)$ because $\mathscr{F}_1(H)e(\varphi_1)$ is simple with center $\mathscr{F}_1$. Moreover I_1 fixes $\mathscr{F}_1$ since $\mathscr{F}_1$ is the scalar

field. But R maps $\mathscr{L}$ onto $\mathscr{F}_1$ since the center is preserved. Thus I_1 fixes $\mathscr{L}$ and $I_1 \subsetneqq I$.

We can now give the analogue of (1.6).

THEOREM 2.8. *The representation of I into $\mathscr{F}(I)e(\theta)$ given by $x \to xe(\theta)$ decomposes into the tensor product of two projective representations over $\mathscr{L}$, U_x and T_x, which map I onto $\mathscr{F}(H)e(\theta)$ and $\mathscr{C}$ respectively where $\mathscr{C}$ is the centralizer of $\mathscr{F}(H)e(\theta)$ in $\mathscr{F}(I)e(\theta)$. The dimensions over $\mathscr{L}$ of these two algebras are $\varphi_1(1)^2$ and a^2 respectively.*

Proof. Each element of I acts by conjugation on $\mathscr{F}(H)e(\theta)$ in such a way that the center is left fixed. Every such automorphism of this simple algebra is inner. Hence for each x in I there is an element U_x in $\mathscr{F}(H)e(\theta)$ such that $x^{-1}wx = U_x^{-1}wU_x$ for every w in $\mathscr{F}(H)e(\theta)$. Clearly the element $\alpha(x, y) = U_xU_yU_{xy}^{-1}$ induces the identity automorphism so $\alpha(x, y)$ is in $\mathscr{L}$ and it follows that $x \to U_x$ is a projective representation of I with factor set α having values in $\mathscr{L}$.

Now let $\mathscr{C}$ denote the centralizer of $\mathscr{F}(H)e(\theta)$ in $\mathscr{F}(I)e(\chi)$. By Theorem 13, page 53 of [1] it follows

$$(2.9) \qquad \mathscr{F}(I)e(\theta) = \mathscr{F}(H)e(\theta) \cdot \mathscr{C} \cong \mathscr{F}(H)e(\theta) \underset{\mathscr{L}}{\otimes} \mathscr{C}$$

because $\mathscr{F}(H)e(\theta)$ and $\mathscr{F}(I)e(\theta)$ both have center $\mathscr{L}$. We also know then that $\mathscr{C}$ is simple with center $\mathscr{L}$. Set $T_x = xU_x^{-1}$ so T_x is in $\mathscr{C}$. Then $x \to T_x$ is a projective representation of I with factor set α^{-1} and $xe(\theta) = U_x \otimes T_x$ as required.

We know $(\mathscr{F}(H)e(\theta): \mathscr{L}) = \varphi_1(1)^2$ and from (2.5) that

$$(\mathscr{F}(I)e(\theta): \mathscr{L}) = (\chi(1)/t)^2 .$$

We also have $\chi(1) = at\varphi_1(1)$ so that we easily obtain from (2.9) the dimension of $\mathscr{C}$ over $\mathscr{L}$ is a^2.

COROLLARY 2.10. *If $\mathscr{F}(\varphi_1) = \mathscr{F}$ then the algebra $\mathscr{F}(G)e(\theta)$ decomposes as the tensor product $\mathscr{F}(H)e(\theta) \otimes \mathscr{C}$ and the representation offered by M decomposes as the tensor product of two projective representations into $\mathscr{F}(H)e(\theta)$ and $\mathscr{C}$ respectively.*

Proof. Since $\mathscr{F}(\varphi_1) = \mathscr{F}$ we have $\mathscr{L} = \mathscr{F}$ and so $G = I$ by (2.6).

3. **The Abelian case.** We continue with the same situation except that we now suppose G/H is *abelian*.

Since I is the subgroup fixing φ_1, there is an absolutely irreducible

character ζ of I such that $\zeta^G = \chi$ and $\zeta \mid H = a\varphi_1$. Now $I \lhd G$ so that in fact we obtain

(3.1) $\chi \mid I = \zeta_1 + \cdots + \zeta_t, \zeta_i \mid H = a\varphi_i$

where the ζ_i are irreducible characters of I conjugate under the action of G and $\zeta = \zeta_1$.

LEMMA 3.2.   *There exists a subgroup J of I containing H and an irreducible character τ of J such that $\zeta \mid J = a\tau$ and $\tau^I = a\zeta$. Then also $\tau \mid H = \varphi_1$ and $[I : J] = a^2$.*

Proof.   Let Λ be the set of linear characters, λ, of I/H such that $\zeta\lambda = \zeta$ and set $J = \cap \ker \lambda$ as λ runs through Λ. We show this J has the required properties.

Let ρ denote the character of the regular representation of I/H. It is a straight-forward computation to verify that $a\varphi_1^I = \zeta\rho$ since both sides are 0 off H and equal to $a[I : H]\varphi_1$ on H. By Frobenius reciprocity ζ has multiplicity a in φ_1^I and so has multiplicity a^2 in $\zeta\rho$. But ρ is the sum of the distinct linear characters of I/H and so there are exactly a^2 linear characters λ such that $\zeta\lambda = \zeta$. Hence $\mid \Lambda \mid = a^2$. Note that Λ is also a subgroup of the group of linear characters of I/H so by the duality theory of abelian groups we obtain $[I : J] = a^2 = \mid \Lambda \mid$.

Now let τ be an irreducible character of J contained in $\zeta \mid J$ with multiplicity b say. If $[I : J]$ is a prime then either $\tau^I = \zeta$ or $\tau^I = \zeta\Psi_1 + \zeta\Psi_2 + \cdots$ where the Ψ_i are the linear characters of I/J. Because I/J is abelian we use induction to find in the general case that τ^I is a sum of characters $\zeta\Psi$ where Ψ is a linear character of I/J. But every such linear character is in Λ so it follows $\tau^I = b\zeta$. Also there is an integer c such that $\tau \mid H = c\varphi_1$ because $\tau \mid H$ is contained in $\zeta \mid H = a\varphi_1$. Now compute degrees of the characters involved.

$$\tau^I(1) = \mid I : J \mid \tau(1) = a^2 c\varphi_1(1) = b\zeta(1) = ba\varphi_1(1) \ .$$

So we obtain $ac = b$.

The decomposition of ζ on J has the form

$$\zeta \mid J = b(\tau + \tau_2 + \cdots + \tau_k)$$

so we find

$$\zeta(1) = a\varphi_1(1) = bk\tau(1) = bkc\varphi_1(1) \ .$$

Thus $a = bkc$ and along with $ac = b$ we find $k = c = 1$ and $a = b$ which proves the lemma.

Notice that $\tau \,|\, H = \varphi_1$ implies that τ has precisely t conjugates under G, $\tau = \tau_1, \cdots, \tau_t$ and the numbering can be arranged to satisfy $\tau_i \,|\, H = \varphi_i$. Then also $\chi \,|\, J = a(\tau_1 + \cdots + \tau_t)$. We shall make use of this in the next result.

LEMMA 3.3. $\mathscr{F}(J)e(\theta)$ *is a simple algebra with center* $\mathscr{L}$.

Proof. The ring $\mathscr{F}(J)e(\theta)$ is semi-simple so simplicity will follow if we show it has only one irreducible module (up to isomorphism). Any irreducible $\mathscr{F}(J)e(\theta)$ module, W, is isomorphic to a direct summand of M_J because M is the unique isomorphism type of $\mathscr{F}(G)e(\theta)$ module. Let μ be the character of W. The character γ must appear in $\mu \,|\, H$ since γ is the only character of an irreducible $\mathscr{F}(H)e(\theta)$ module in M_H. Thus $\mu \,|\, H$ contains each φ_i. Moreover the absolutely irreducible characters in μ must appear in $\chi \,|\, J$. By the remark above the lemma, every τ_i appears in μ. Thus μ is invariant under G and it follows $M_J = k \cdot W$ for some k. Hence $\mathscr{F}(J)e(\theta)$ has only one irreducible module. We find also that $\mathscr{F}(J)e(\theta)$ is isomorphic to $\mathscr{F}(J)e(\mu)$ and its center is isomorphic to $\mathscr{F}(\tau)$. The equations $\zeta \,|\, J = a\tau$ and $\tau^I = a\zeta$ imply $\mathscr{F}(\tau) = \mathscr{F}(\zeta)$. But then $\mathscr{F}(\zeta)$ is isomorphic to the center of $\mathscr{F}(I)e(\theta)$ so by (2.5) and (2.2) $\mathscr{F}(\zeta) \cong \mathscr{F}(\varphi_1)$. These are isomorphisms over $\mathscr{F}$ so in fact $\mathscr{F}(\zeta) = \mathscr{F}(\varphi_1)$ because both are normal extensions. Hence the center of $\mathscr{F}(J)e(\theta)$ is isomorphic to $\mathscr{L}$. Because of the inclusions

$$\mathscr{L} \cong \mathscr{F}(H)e(\theta) \cong \mathscr{F}(J)e(\theta)$$

and the fact that $\mathscr{L}$ centralizes $\mathscr{F}(J)e(\theta)$ we have $\mathscr{L} = $ center $\mathscr{F}(J)e(\theta)$.

Now that we know $\mathscr{F}(I)e(\theta)$ and $\mathscr{F}(J)e(\theta)$ have the same centers we obtain a decomposition

$$(3.4) \qquad \mathscr{F}(I)e(\theta) = \mathscr{F}(J)e(\theta) \cdot \mathscr{C}_J \cong \mathscr{F}(J)e(\theta) \otimes \mathscr{C}_J$$

where $\mathscr{C}_J$ is the centralizer of $\mathscr{F}(J)e(\theta)$. If we apply (2.8) with J in place of H we find dimension $\mathscr{C}_J$ over $\mathscr{L}$ is a^2. This is the same a that appears for H because $\chi \,|\, J = a(\tau_1 + \cdots + \tau_t)$. It is clear that $\mathscr{C}$ (the centralizer of $\mathscr{F}(H)e(\theta)$) contains $\mathscr{C}_J$ and so by dimension count we find $\mathscr{C}_J = \mathscr{C}$. From this it follows that $\mathscr{F}(J)e(\theta) = \mathscr{F}(H)e(\theta)$. This makes it possible to adjust the projective representations U and T so that $U_x = xe(\theta)$ if x is in J and T is constant on the cosets of J.

We are now able to identify the algebra $\mathscr{C}$.

PROPOSITION 3.5.   *Let* $\mathscr{F}_1$ *denote* $\mathscr{F}(\varphi_1)$. *The algebra* $\mathscr{C}$ *is iso-*

morphic to a twisted group alegebra $\mathscr{F}_1(I/J)_\beta$ *for some factor set* β
on I/J *with values in* $\mathscr{F}_1$.

Proof. Recall that the twisted group algebra $\mathscr{F}_1(I/J)_\beta$ has a
basis $\{t_x \mid x \in I/J\}$ such that $t_x t_y = \beta(x, y)t_{xy}$. The modification of U
above allows us to view T as a projective representation of I/J. Then
the correspondence $t_x \to T_x$ for x in I/J induces a homomorphism from
$\mathscr{F}_1(I/J)_\beta$ into $\mathscr{C}$ provided we have fixed identification of $\mathscr{F}_1$ with
$\mathscr{L}$ and $\beta = \alpha^{-1}$ on I/J. If we show this homomorphism is onto $\mathscr{C}$,
we will be finished because both algebras have dimension a^2 over $\mathscr{F}_1$.
From equation (3.4) it follows

$$\mathscr{F}(I)e(\theta) = \sum_{x \in I/J} \mathscr{F}(J)e(\theta) \otimes T_x$$

becase the right side contains $\mathscr{L}$ along with every $ye(\theta)$ for y in I.
It follows that the T_x span $\mathscr{C}$ over $\mathscr{L}$ (because the tensor product
is taken over $\mathscr{L}$) and hence the homomorphism above is onto $\mathscr{C}$.

Let A denote I/J. The fact that $\mathscr{F}_1(A)_\beta$ is simple with center
$\mathscr{F}_1$ imposes restrictions on A and one can say quite a lot about $\mathscr{F}_1(A)_\beta$.
We shall give a brief sketch of the results of DeMeyer [3] which are
relevant.

Consider the function $\eta(a, b) = \beta(a, b)/\beta(b, a)$. Because A is abelian
and β is a factor set, it follows that η is a (multiplicative) skew
bilinear form from $A \times A$ to the multiplicative group of $\mathscr{F}_1$. This is
$\eta(ab, c) = \eta(a, c)\eta(b, c)$ and $\eta(a, b) = \eta(b, a)^{-1}$. Because $\mathscr{F}_1$ is the center
of $\mathscr{F}_1(A)_\beta$, η is nondegenerate; that is $\eta(a, A) = 1$ holds only for $a = 1$. In a way similar to the method of decomposing a vector space
admitting a skew bilinear form, one decomposes A into the direct sum
of "hyperbolic planes". The result is the following.

THEOREM (*DeMeyer* [3]). *Let* $\mathscr{F}_1(A)_\beta$ *be central simple over* $\mathscr{F}_1$.
Then A *decomposes as*

$$A = (C_{11} \times C_{12}) \times (C_{21} \times C_{22}) \times \cdots \times (C_{r1} \times C_{r2})$$

where C_{ij} *is cyclic of prime-power order and* $C_{i1} \cong C_{i2}$. *The function*
η *remains nondegenerate on* $C_{i1} \times C_{i2}$ *and the subalgebras* $\mathfrak{A}_i =$
$\mathscr{F}_1(C_{i1} \times C_{i2})_\beta$ *are central simple over* $\mathscr{F}_1$. *Finally we have the decomposition*

$$\mathscr{F}_1(A)_\beta \cong \mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_r .$$

This decomposition allows us to get information about the index
of the algebra $\mathscr{F}_1(A)_\beta$. We must now restrict $\mathscr{F}$ to be an *algebraic
number field*. Now the Brauer-Hasse-Noether theorem can be applied.

It tells us the index of a finite dimensional division algebra over $\mathscr{F}$ is equal to its exponent; that is the order of its class in the Brauer group of the center of the division algebra.

The index of each $\mathfrak{A}_i$ is a divisor of $|C_{i1}|$ since dimension of $\mathfrak{A}_i$ over $\mathscr{F}_1$ is $|C_{i1}|^2$. Thus the index of $\mathscr{F}_1(A)_\beta$ divides the least common multiple of the indices of the $\mathfrak{A}_i$ since the exponent of $\mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_r$ divides the l.c.m. of the exponents of the $\mathfrak{A}_i$. This in turn divides the l.c.m. of the numbers $|C_{i1}| \, 1 \leq i \leq r$ and this number is precisely the exponent of A. So we have the

PROPOSITION 3.6.   *The index of the algebra $\mathscr{C}$ divides the exponent of I/J.*

THEOREM 3.7.   *The Schur index $m_{\mathscr{F}}(\chi)$ of χ divides*

$$[G : I] \cdot \text{l.c.m.}\{m_{\mathscr{F}}(\varphi_1), \quad \text{exponent} \quad (I/J)\} \, .$$

Proof.   We have $\mathscr{F}_1 \otimes \mathscr{F}(G)e(\theta)$ equivalent to $\mathscr{F}(I)e(\theta)$ in the Brauer group of $\mathscr{F}_1$ because by Theorem 16, page 56 of [1],

$$\mathscr{F}_1 \otimes \mathscr{F}(G)e(\theta) \cong \mathscr{M}_t(\mathscr{F}) \otimes_{\mathscr{F}} \mathscr{F}(I)e(\theta) \, .$$

Hence by Theorem 20, page 59 of [1] the factor by which the index has been reduced after extending the field to $\mathscr{F}_1$ must divide $[\mathscr{F}_1 : \mathscr{F}]$. By (2.6) this number is $[G : I]$ so the index of $\mathscr{F}(G)e(\theta)$ divides $[G : I]$ times the index of $\mathscr{F}(I)e(\theta)$. By the decomposition of (2.9) we see the index of $\mathscr{F}(I)e(\theta)$ divides the least common multiple of the index of $\mathscr{F}(H)e(\theta)$ and the index of $\mathscr{C}$. The index of $\mathscr{F}(H)e(\theta)$ is $m_{\mathscr{F}}(\varphi_1)$ so the result follows from (3.6).

There is a theorem of Brauer [2, Theorem 70.28] which shows that certain questions about the Schur index of an irreducible character for a finite group can be reduced to questions about $\mathscr{F}$-elementary groups. Recall that among other things an $\mathscr{F}$-elementary group is a semi-direct product PA of a p-group P and a normal cyclic p'-group A. We now consider the case where $G = PA$ is such a semi-direct product (not necessarily $\mathscr{F}$-elementary however). Let $H = C_G(A)$ so that H is normal and G/H is abelian (because A has an abelian automorphism group). Let χ be an absolutely irreducible character of G. In this situation we have the following.

THEOREM 3.8.   *Assume $\mathscr{F}$ is an algebraic number field. Then there exists a chain of normal subgroups*

$$G \supseteq G_1 \supseteq I \supseteq J \supseteq H$$

such that the Schur index of χ over $\mathscr{F}$ divides $2[G_1: I]\cdot$ exponent (I/J). The factor 2 can be omitted if $p \neq 2$ or if $p = 2$ but $I \neq J$.

Proof. Since the Schur index will not change, assume $\mathscr{F} = \mathscr{F}(\chi)$. Let M_0 be an irreducible $\mathscr{F}(G)$-module with character $\theta_0 = m\chi$. Suppose $M_0 \mid H = f(V_1 \oplus \cdots \oplus V_r)$ with the V_i distinct irreducible $\mathscr{F}(H)$-modules. Let $G_1 = \{x \in G \mid xV_1 \cong V_1 \text{ as } \mathscr{F}(H)\text{-modules}\}$. Then there is an irreducible $\mathscr{F}(G_1)$-module W_1 such that $W_1^G \cong M_0$ and $W_1 \mid H \cong f\cdot V_1$. Since G/H is abelian we know $G_1 \lhd G$ so $M_0 \mid G_1 = W_1 \oplus \cdots \oplus W_r$ where the W_i are mutually nonisomorphic irreducible $\mathscr{F}(G_1)$-modules. Now for any nonzero δ in End$_{\mathscr{F}(G)}(M_0)$ we have $\delta(W_1) \cong W_1$ and since the W_i are nonisomorphic, $\delta(W_1) = W_1$. Hence we imbed End$_{\mathscr{F}(G)}(M_0)$ into End$_{\mathscr{F}(G_1)}(W_1)$. Conversely the equation $W_1^G \cong M_0$ provides a natural imbedding of End$_{\mathscr{F}(G_1)}(W_1)$ into End$_{\mathscr{F}(G)}(M_0)$. Hence these two division algebras are isomorphic. Let $\chi \mid G_1$ contain the character ζ which also appears in the decomposition of the character for W_1. We have $m_{\mathscr{F}}(\chi) = m_{\mathscr{F}}(\zeta)$ since these numbers represent the indices of the respective endomorphism rings above. Moreover $\mathscr{F}(\zeta)$ is the center of End$_{\mathscr{F}(G_1)}(W_1)$ so $\mathscr{F} = \mathscr{F}(\zeta)$. We may now apply (3.7) to $\mathscr{F}$, G_1, and W_1 in place of $\mathscr{F}, G, M$ and obtain

$$m_{\mathscr{F}}(\chi) \mid [G_1: I] \text{ l.c.m. } \{\text{exponent } (I/J), m_{\mathscr{F}}(\varphi_1)\}$$

where again φ_1 is an irreducible character of H contained in $\chi \mid H$. But $H = C_P(A) \times A$ is a nilpotent group so by Roquette's theorem [6], $m_{\mathscr{F}}(\varphi_1) = 1$ or possibly 2 in case $p = 2$. Even when $m_{\mathscr{F}}(\varphi_1) = 2$ the l.c.m. of exponent (I/J) and $m_{\mathscr{F}}(\varphi_1)$ will be exponent (I/J) provided $I \neq J$. So the result follows.

COROLLARY 3.9. *If $\chi \mid H = a(\varphi_1 + \cdots + \varphi_t)$ then $m_{\mathscr{F}}(\chi) \mid 2at$ where the 2 can be omitted if $p \neq 2$.*

Proof. If $\chi \mid H = a(\varphi_1 + \cdots + \varphi_t)$ then $\zeta \mid H = a(\varphi_1 + \cdots + \varphi_r)$ where $r \mid t$. Thus $\mid I: J \mid = a^2$ and exponent (I/J) divides a. The definition of I yields $[G_1: I] = r$ so $[G_1: I] \mid t$. The result now follows from the theorem.

REMARKS. (a) It can happen that $m_{\mathscr{F}}(\chi) = at$. This is the case when χ is an irreducible character of degree 3 for the metacyclic group $\langle x, y \rangle$ where $x^7 = y^9 = 1$ and $y^{-1}xy = x^4$.

(b) The application of DeMeyer's theorem shows the interest in twisted group algebras, $\mathscr{F}(G)_\alpha$, which are simple with center $\mathscr{F}$. A discussion of groups G which admit such a factor set α can be found in [4].

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