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COMPACT SOBOLEV IMBEDDINGS FOR UNBOUNDED DOMAINS

ROBERT A. ADAMS

A condition on an open set $G \subset E_n$ which is both necessary and sufficient for the compactness of the (Sobolev) imbedding $H_0^{m+1}(G) \rightarrow H_0^m(G)$ is not yet known. C. Clark has given a necessary condition (quasiboundedness) and a much stronger sufficient condition. We show here that (unless $n=1$) quasiboundedness is not sufficient, and answer in the negative a question raised by Clark on whether the imbedding can be compact if ∂G consists of isolated points. We also substantially weaken Clark's sufficient condition so as to include a wide class of domains with null exterior. The gap between necessary and sufficient conditions is thus considerably narrowed.

Let G be an open set in Euclidean n-space, E_n. Let $H_0^m(G)$ for each nonnegative integer m denote the Sobolev space obtained by completing with respect to the norm

$$\| u \|_{m,G} = \left\{ \sum_{|\alpha| \leq m} \int_G | D^\alpha u(x)|^2 \, dx \right\}^{1/2}$$

the space $C_0^\infty(G)$ of all infinitely differentiable complex valued functions having compact support in G. Here, as usual, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers; $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $D_j = \partial/\partial x_j$, $j = 1, \cdots, n$.

We shall say that G has the Rellich property if for each integer $m \geqslant 0$ the imbedding mapping $H_0^{m+1}(G) \rightarrow H_0^m(G)$ is compact. It is well known that any bounded G has this property. An unbounded domain G is called quasibounded if dist $(x, \partial G) \rightarrow 0$ whenever x tends to infinity in G. If G is unbounded and not quasibounded then it contains an infinite number of mutually disjoint, congruent balls. If φ is infinitely differentiable, has support in one of these balls, and has nonzero $L^2(G)$ norm then the set of its translates with supports in the other balls provides a counterexample showing the imbedding $H_0^1(G) \rightarrow H_0^0(G) \equiv L^2(G)$ is not compact. Thus for an unbounded domain quasiboundedness is necessary for the Rellich property.

In [2] Clark showed that the following Condition 1 is sufficient to guarantee that G has the Rellich property.

CONDITION 1. To each $R \geqq 0$ there correspond positive numbers $d(R)$ and $\delta(R)$ satisfying

(a) $d(R) + \delta(R) \rightarrow 0$ as $R \rightarrow \infty$,

(b) $d(R)/\delta(R) \leqq M < \infty$ for all R,

(c) for each $x \in G$ with $|x| > R$ there exists y such that $|x-y| < d(R)$ and $G \cap \{z: |z - y| < \delta(R)\} = \emptyset$.

This condition is considerably stronger than quasiboundedness. It implies, for example, that G has nonnull exterior. In [3] Clark gave an example of an unbounded domain having the Rellich property but not satisfying Condition 1. His example was the "spiny urchin," an open connected set in E_2 obtained by removing from the plane all points whose polar coordinates (r, θ) satisfy for any $k = 1, 2, \cdots$ the two restrictions $r \geqq k$ and $\theta = 2^{-k}m\pi$, $m = 1, 2, \cdots, 2^{k+1}$.

In this paper the gap between quasiboundedness as a necessary condition and Condition 1 as a sufficient condition for a domain to have the Rellich property is narrowed from both ends. On the one hand we show that if $n \geqq 2$ then no open set whose boundary consists only of isolated points with no finite accumulation point can have the Rellich property. This settles a question raised by Clark in [3]. On the other hand we show that Condition 1 can be replaced by the following weaker Condition 2, which is still sufficient to guarantee that G has the Rellich property. In the statement $B_r(x)$ denotes the open ball of radius r about x.

CONDITION 2. There exists $R_0 \geqq 0$ such that to each $R \geqq R_0$ there correspond numbers $d(R), \delta(R) > 0$ such that

(a) $d(R) + \delta(R) \rightarrow 0$ as $R \rightarrow \infty$,

(b) $d(R)/\delta(R) < M \leqq \infty$ for all $R \geqq R_0$,

(c) for each $x \in G$ such that $|x| > R \geqq R_0$ the ball $B_{3d(R)}(x)$ is disconnected into two open components C_1 and C_2 by an $n-1$ dimensional manifold forming part of the boundary of G in such a way that each of the two open sets $C_i \cap B_{d(R)}(x)$, $i = 1, 2$, contains a ball of radius $\delta(R)$.

Roughly speaking if the $n - 1$ dimensional manifolds in the boundary of G are reasonably smooth and unbroken, and bound a quasibounded domain (containing G) then G will satisfy Condition 2. Clark's "spiny urchin" is an example of such a domain. If $n = 1$ any quasibounded domain satisfies Condition 2, (but not necessarily Condition 1) and so in this case quasiboundedness is necessary and sufficient for the Rellich property.

Our principal results are as follows

THEOREM 1. *If G is open in E_n, $n \geqq 2$, and the boundary of G consists only of isolated points with no finite accumulation point, then the imbedding $H_0^1(G) \rightarrow L^2(G)$ is not compact. Thus quasibounded-*

ness is not sufficient to guarantee the Rellich property.

THEOREM 2. *If G is open in E_n and satisfies Condition 2 then it has the Rellich property.*

For the proof of Theorem 1 we require the following

LEMMA 1. *Given $\rho, \delta > 0$, $x_0 \in E_n$ $(n \geqq 2)$, there exists a function $u \in C^\infty(E_n)$ with the following properties*

(1) $u(x) = 0$ *in a neighbourhood of* x_0

(2) $0 \leqq u(x) \leqq 1$ *for all* x

(3) $u(x) = 1$ *outside the ball* $B_\rho(x_0)$

(4) $\int_{En} |\nabla u(x)|^2 dx^2 \leqq \delta^2$.

Proof. Let $f \in C^\infty(R)$ satisfy $0 \leqq f(t) \leqq 1$, $f(t) = 1$ for $t \geqq 1$ and $f(t) = 0$ in a neighbourhood of $t = 0$. Let m be a positive integer, put $r = |x - x_0|$ and define

$$u(x) = v(r) = f([r/\rho]^{1/m}) .$$

Clearly $u \in C^\infty(E_n)$ and satisfies (1), (2) and (3). Also

$$|\nabla u(x)|^2 = \sum_{i=1}^n |D_i u(x)|^2 = |v'(r)|^2 .$$

Denoting by ω_n the surface area of the unit sphere in E_n and making the change of variables $t = (r/\rho)^{1/m}$ we obtain

$$\begin{aligned} \int_{En} |\nabla u(x)|^2 dx &= \omega_n \int_0^\rho \left| \frac{d}{dr} f\left(\left[\frac{r}{\rho}\right]^{1/m}\right)\right|^2 r^{n-1} dr \\ &= \omega_n \rho^{n-2} m^{-1} \int_0^1 \left|\frac{d}{dt} f(t)\right|^2 t^{1+m(n-2)} dt \\ &\leqq \omega_n \rho^{n-2} m^{-1} [2 + m(n-2)]^{-1} \sup_{0 \leqq t \leqq 1} |f'(t)|^2 \end{aligned}$$

which, for $n \geqq 2$, can be made less than δ^2 for a suitably large choice of m.

REMARK. If $\varphi \in C_0^\infty(E_n)$ and u is constructed as above, then $\varphi \cdot u \in C_0^\infty(E_n - \{x_0\}) \subset H_0^1(E_n - \{x_0\})$.

Proof of Theorem 1. Let Q be a fixed open ball in E_n. Let $\varphi \in C_0^\infty(Q)$ be extended to all of E_n so that $\varphi(x) = 0$ in $E_n - Q$. Suppose $\varphi(x) \geqq 0$ for all x and

$$||\varphi||_{0,E_n} = C > 0 , \qquad ||\varphi||_{1,E_n} = K > 0 .$$

There exists $M > 0$ such that for all x in E_n

$$|\varphi(x)| \leqq M\,, \qquad |D_j\varphi(x)| \leqq M\,, \qquad j = 1, \cdots, n\,.$$

If Q contains no boundary points of G put $\psi = \varphi$. Otherwise Q contains only a finite number of boundary points of G, say $x_1, \cdots, x_k$. For $i = 1, \cdots, k$ let $B_i = B_{\rho_i}(x_i)$ where ρ_i is small enough that vol. $B_i \leqq (C/2kM)^2$. Let $\delta = K/Mk$ and let u_i be the function constructed as in Lemma 1 corresponding to the point x_i and the constants ρ_i and δ. Put $\psi = \varphi \cdot u_1 \cdots u_k$. Clearly $\psi \in H_0^1(Q - \{x_i, \cdots, x_k\}) \subset H_0^1(G)$. We have

$$\begin{aligned} \|\psi\|_{0,G} &\geqq \|\varphi\|_{0,E_n} - \sum_{i=1}^{k} \|\varphi\|_{0,B_i} \\ &\geqq C - \sum_{i=1}^{k} M(\text{vol. } B_i)^{1/2} \geqq \frac{1}{2}C\,. \end{aligned}$$

Also

$$\begin{aligned} \|D_j\psi\|_{0,G} &\leqq \|D_j\varphi\|_{0,E_n} + \sum_{i=1}^{k} \|\varphi u_1 \cdots D_j u_i \cdots u_k\|_{0,B_i} \\ &\leqq K + kM\delta = 2K\,. \end{aligned}$$

Since $\|\psi\|_{0,G} \leqq \|\varphi\|_{0,G} = C$ we have

$$\|\psi\|_{1,G} \leqq (C^2 + 4nK^2)^{1/2} = C_1\,.$$

Now let $\{Q_i\}_{i=1}^{\infty}$ be a family of mutually disjoint open balls in E^n all congruent to Q. Let φ_i be a translate of φ with support in Q_i and let $\psi_i \in H_0^1(G)$ be constructed from φ_i as above, so that

$$\|\psi_i\|_{0,G} \geqq \frac{C}{2}\,, \qquad \|\psi_i\|_{1,G} \leqq C_1\,.$$

Then the sequence $\{\psi_i\}_{i=1}^{\infty}$ is bounded in $H_0^1(G)$ but contains no subsequence convergent in $L^2(G)$ since for $i \neq j$ $\|\psi_i - \psi_j\|_{0,G} \geqq C/\sqrt{2}$. Thus the imbedding $H_0^1(G) \to L^2(G)$ is not compact.

The proof of Theorem 2 is based on the following generalization of Poincaré's inequality which is a variant on those forms appearing in Agmon [1] and Clark [2].

LEMMA 2. *Let G be open in E_n and satisfy Condition 2. Let G_R denote $G \cap \{x: |x| > R\}$. Then there exists a constant c depending only on n and M (the constant of Condition 2 (b)) such that for all $R \geqq R_0$ and every $u \in H_0^1(G)$*

$$\int_{G_R} |u(x)|^2\, dx \leqq c(d(R))^2 \int_G |\nabla u(x)|^2\, dx\,.$$

Proof. Fix $R \geqq R_0$ and let $d = d(R)$, $\delta = \delta(R)$. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of integers let $Q_\alpha = \{x \in E_n : \alpha_k n^{-1/2} d \leqq x_k \leqq (\alpha_k + 1) n^{-1/2} d\}$. Then $E_n = \bigcup_\alpha Q_\alpha$. Let $\varphi \in C_0^\infty(G)$. Fix $x \in G_R$. Then $x \in Q_\alpha$ for some α. Let $B_d = B_d(x)$, $B_{3d} = B_{3d}(x)$. There exists an $n-1$ dimensional manifold forming part of ∂G which disconnects B_{3d} into open components C_1 and C_2 and there exist points $y_i \in C_i$ $(i = 1, 2)$ such that $B_\delta(y_i) \subset C_i$. Thus φ can be written as $\varphi = \varphi_1 + \varphi_2$ where $\varphi_i \in C_0^\infty(G)$ and $\varphi_1 \equiv 0$ in C_2 while $\varphi_2 \equiv 0$ in C_1. Since $Q_\alpha \subset B_d$ we have

$$\int_{Q_\alpha \cap G_R} |\varphi(y)|^2 dy \leqq \int_{C_1 \cap B_d} |\varphi_1(y)|^2 dy + \int_{C_2 \cap B_d} |\varphi_2(y)|^2 dy .$$

If (r, σ) and S denote respectively spherical coordinates in E_n centered at y_2 and the surface of the unit sphere about y_2 we have

$$\begin{aligned} \int_{C_1 \cap B_d} |\varphi_1(y)|^2 dy &\leqq \int_S d\sigma \int_\delta^{2d} |\varphi_1(r, \sigma)|^2 r^{n-1} dr \\ &\leqq 2d \int_S |\varphi_1(t, \sigma)|^2 t^{n-1} d\sigma \end{aligned}$$

where $t = t(\sigma)$ satisfies $\delta \leqq t \leqq 2d$. Since $\varphi_1(\delta, \sigma) = 0$ it follows that

$$\begin{aligned} |\varphi_1(t, \sigma)^2 t^{n-1}| &= \left| \int_\delta^t \frac{d}{dr} \varphi_1(r, \sigma) dr \right|^2 t^{n-1} \\ &\leqq (2d)^n \int_\delta^{2d} \left| \frac{d}{dr} \varphi_1(r, \sigma) \right|^2 dr \\ &\leqq (2d)^n \delta^{1-n} \int_\delta^{2d} \left| \frac{d}{dr} \varphi_1(r, \sigma) \right|^2 r^{n-1} dr . \end{aligned}$$

Thus, since $d/\delta < M$,

$$\begin{aligned} \int_{C_1 \cap B_d} |\varphi_1(y)|^2 dy &\leqq (2d)^{n+1} \delta^{1-n} \int_S d\sigma \int_\delta^{2d} \left| \frac{d}{dr} \varphi_1(r, \sigma) \right|^2 r^{n-1} dr \\ &\leqq 2^{n+1} M^{n-1} d^2 \int_{\delta \leqq |y - y_2| \leqq 2d} |\nabla \varphi_1(y)|^2 dy \\ &\leqq 2^{n+1} M^{n-1} d^2 \int_{B_{3d}} |\nabla \varphi_1(y)|^2 dy . \end{aligned}$$

Combining this with a similar expression for φ_2 we obtain

$$\begin{aligned} \int_{Q_\alpha \cap G_R} |\varphi(y)|^2 dy &\leqq 2^{n+1} M^{n-1} d^2 \int_{B_{3d}} |\nabla \varphi(y)|^2 dy \\ &\leqq 2^{n+1} M^{n-1} d^2 \int_{Q'_\alpha} |\nabla \varphi(y)|^2 dy \end{aligned}$$

where Q'_α is the union of all the sets Q_α which intersect B_{3d}. There is a number N depending only on n such that any $N+1$ of the sets Q'_α have null intersection. Summing the above inequality over all α for which Q_α intersects G_R we obtain

$$\int_{G_R} |\varphi(y)|^2\,dy \leqq 2^{n+1}NM^{n-1}(d(R))^2 \int_G |\nabla\varphi(y)|^2\,dy\ .$$

This inequality extends by completion to $H_0^1(G)$.

The remaining part of the proof of Theorem 2 is similar to Clark's proof [2, Th. 3] and is included here for completeness. First, however, let $H^m(G, R)$ be the completion in the norm $\|\cdot\|_{m,G\cap K_R}$ of the space $C_0^\infty(G, R)$ of all C^∞ functions whose support is a compact subset of $G \cap K_R$ where $K_R = \overline{B_R(0)}$. Since the imbedding $H_0^{m+1}(K_R) \to H_0^m(K_R)$ is known to be compact [4, Chapter XIV] and since an element of $H^m(G, R)$ can be extended to be zero outside its support so as to belong to $H_0^m(K_R)$ it follows that the imbeddings $H^{m+1}(G, R) \to H^m(G, R)$, $m = 0, 1, 2, \cdots$ are compact.

Proof of Theorem 2. It suffices, by an inductive argument, to prove only that the imbedding $H_0^1(G) \to L^2(G)$ is compact. We make use of the following well known compactness criterion for sets in $L^2(G)$: if $G \subset E_n$ and the sequence $\{u_k\}_{k=1}^\infty$ is bounded in $L^2(G)$ then it is compact in $L^2(G)$ provided

(a) for every bounded $G' \subset G$ the sequence $\{u_k \mid G'\}$ is compact in $L^2(G')$, and

(b) for each $\varepsilon > 0$ there exists $R > 0$ such that for all k

$$\int_{G_R} |u_k(x)|^2\,dx < \varepsilon\ .$$

Now let $\{u_k\}$ be a sequence bounded in $H_0^1(G)$, say $\|u_k\|_{1,G} \leqq K$. By Lemma 2, for $R \geqq R_0$ we have $\|u_k\|_{0,G_R} \leqq C(d(R))^2K \to 0$ as $R \to \infty$ so condition (b) of the criterion is satisfied. To establish (a) let G' be a bounded subset of G, so that $G' \subset K_R$ for some R. Since $\{u_k \mid K_R\}$ is bounded in $H^1(G, R)$ it is compact in $H^0(G, R) = L^2(K_R \cap G)$ and so $\{u_k \mid G'\}$ is compact in $L^2(G')$. Thus $\{u_k\}$ is compact in $L^2(G)$, whence the theorem.

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THE UNIVERSITY OF BRITISH COLUMBIA