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PREASSIGNING THE SHAPE OF A FACE

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THEOREM. Let G be a 3-connected planar graph, F a nonseparating n-circuit of G with nodes $A_1, \cdots, A_n$ (in a cyclic order of F), and let F' be a convex n-gon with vertices $A'_1, \cdots, A'_n$ (in cyclic order). Then there exists a 3-polytope P which realizes G, such that F' is a face of P and that the vertex A'_i of P corresponds to the node A_i of G for $i = 1, 2, \cdots, n$.

A remarkable theorem of Steinitz (see [2, p. 77], [3, p. 192], [1, p. 235]) asserts that a graph G is realizable by a 3-polytope (i.e., isomorphic to the graph of the vertices and edges of a 3-dimensional convex polytope P) if and only if G is planar and 3-connected. Moreover, this realization is unique up to combinatorial equivalence of the 3-polytopes, the boundaries of the 2-faces of each polytope realizing G being determined by nonseparating circuits in G. (The regions of the plane determined in any imbedding of G in the plane by the nonseparating circuits of G will be called *faces* of G.)

Combinatorially equivalent polytopes may—intuitively speaking—have various shapes. Among the different problems concerning the freedom of choice of 3-polytopes realizing a given graph G, one may ask whether the shape of one n-gonal 2-face F of P can be required to be any preassigned (convex) n-gon F' (see [1, p. 244]). The solution of this problem is trivially affirmative if F is a triangle or a quadrangle, since in this case there exists an affine, respectively permissible for P projective transformation of any realization P of G carrying F onto any predetermined F'. However, the problem ceases to be trivial already for a pentagonal face F.

It is the aim of the present note to establish the affirmative solution of the above problem for any face F. More precisely, we shall prove the theorem enunciated at the outset. By the remark made above, we shall without loss of generality assume that $n \geqq 5$.

The proof of the theorem consists mainly of a repetition of the proof of Steinitz's theorem as given in [1, pp. 236–242], with modifications resulting from the desire to interfere with F as little as possible and, if such interference is unavoidable, to conduct it under special precautions. Since a repetition of the whole argument would lead to a needless duplication, we shall restrict ourselves to a summary of the proof of Steinitz's theorem, indicating the main ideas and the necessary changes in the argumentation. We shall use the terminology and notation employed in [1].

The proof of Steinitz's theorem proceeds by induction on the number of edges of G, and two cases have to be considered. In each case, the graph G is "reduced" to a 3-connected planar graph G^* by a change of such a nature that from each realization of G^* by a 3-polytope P^* a realization of G by a 3-polytope P may be constructed. The "reductions" needed for the proof are indicated in Figure 1.

Case 1. The graph contains a nonseparating circuit with three edges (i.e., a triangular face, or *triangle*) one node of which has valence 3. Then one of the reductions η_1, η_2, η_3 may be applied, yielding a G^* with 1, 2, or 3 edges less than G. In the proof of Steinitz's theorem, there is nothing to be added in this case, since induction takes over. In the proof of our theorem one has to require that G^* be realized by a 3-polytope P^* having F' as a face, unless the circuit F contains one of the edges—marked by one, two, or three stars in Figure 1—of the triangle in question. In those exceptional cases the circuit F^* of G^* (corresponding to the circuit F of G, and having also n edges in cases denoted by one star, $n-1$ edges in cases denoted by two stars, and $n+1$ edges in cases denoted by three stars) has to correspond in P^* to a face $F^{*\prime}$ with vertices $A_1^{*\prime}, A_2^{*\prime}, \cdots$ different from F'. The construction of the polygon $F^{*\prime}$ from F' is indicated

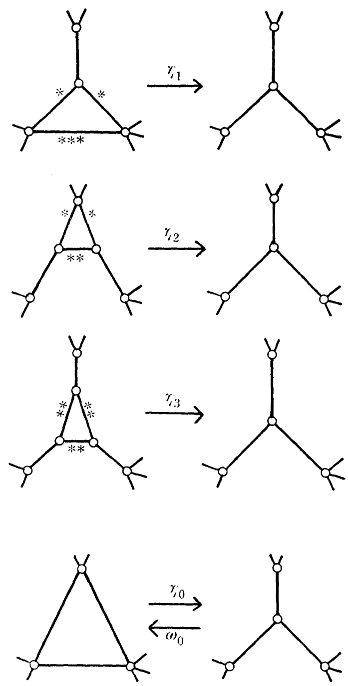

FIGURE 1

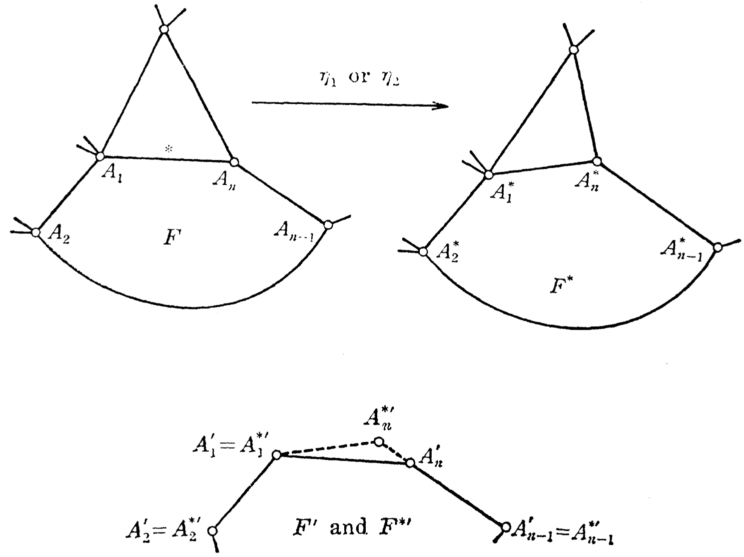

FIGURE 2

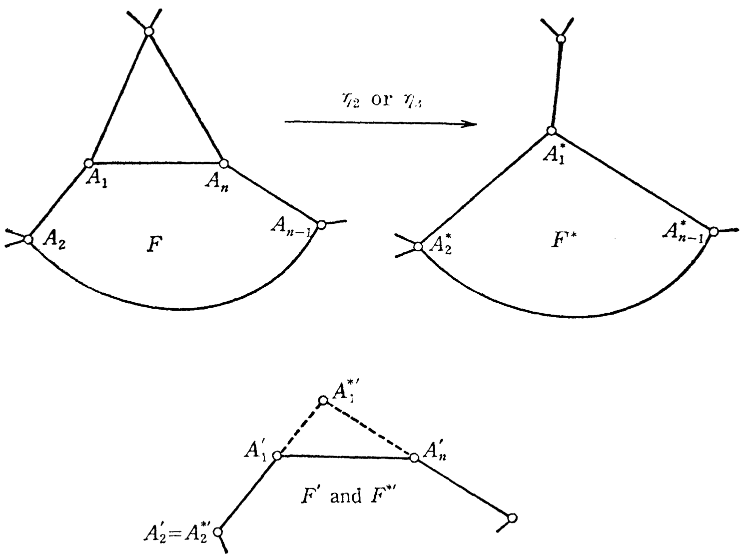

FIGURE 3

in Figures 2 and 3 for the first two cases; in case of the edge denoted by three stars, $F^{*\prime}$ may be any $(n + 1)$-gon obtained as the convex hull of F' and a point $A^{*\prime}_{n+1}$ near the edge $A'_nA'_1$. (Note that in the case represented in Figure 3 it may be necessary to start not with F' itself but—in order to guarantee the existence of $F^{*\prime}$—with a suitable projective transformation of F'. However, this does not impair the construction since a suitable inverse projective transformation of the polytope realizing G will restore F' as a face of P'.)

This completes the proof in Case 1.

Case 2. Now we assume that G contains no triangle possessing at least one 3-valent node. Since in any case Euler's relation implies that G contains either a triangle, or a 3-valent vertex, it follows that now one of the reductions ω_0 or η_0 may be applied to G. However, those reductions do not reduce the number of edges and may thus be deemed useless. The depth of Steinitz's proof lies in establishing that by a judicious choice of a finite sequence of ω_0 or η_0 reductions (not exceeding in total number the number of edges of G) one may transform G into a graph $\tilde{G}$ which is covered by Case 1. In order to prove our theorem we shall show that there is enough freedom in the choice of the ω_0 and/or η_0 reductions to reach $\tilde{G}$ without interfering (in any of the stages) with the circuit F of G (or the corresponding circuits in the intermediate stages). The derived realization of $\tilde{G}$ and of all the intermediate graphs will still have F' as the preassigned polygon.

Before proving those assertions, we have to introduce a number of notions. For each 3-connected planar graph G, we define a graph $I(G)$ as follows. The vertices of $I(G)$ correspond to edges of G; two vertices of $I(G)$ are connected by an edge if and only if the corresponding edges of G have a common node and belong to the same nonseparating circuit. (In Figure 4, a graph G is shown by heavy lines, while $I(G)$ is indicated by thin lines.) It is easily seen that $I(G)$ if planar and 3-connected. Note that $I(G)$ is 4-valent, and that each n-gonal region of $I(G)$ corresponds either to an n-gonal region of G or to an n-valent vertex of G. A "geodesic arc" in $I(G)$ is a path in $I(G)$ in which each two adjacent edges separate the other two edges of $I(G)$ issuing from their common node. A "lens" of $I(G)$ is

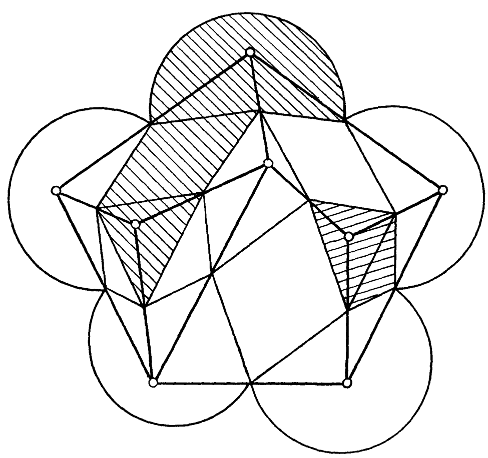

FIGURE 4

a region of plane having as boundary two geodesic arcs of $I(G)$, such that at their common endpoints (the "poles" of the lens) the remaining two edges of $I(G)$ do not belong to the lens. (For example, each of the shaded regions in Figure 4 is a lens, one with poles A and B, the other with poles C and D; the second lens is clearly the simplest possible type, consisting of only two triangles; we shall denote it by L_0.)

An important role is played by "irreducible lenses", that is, lenses which do not contain any proper sublenses. The relevant facts about irreducible lenses are (see [1, pp. 239–242]):

(1) Each region of the plane which has a boundary consisting either of a (closed) geodesic arc, or of two geodesic arcs, contains an irreducible lens. This implies, among other consequences, the existence of irreducible lenses in each $I(G)$.

(2) Each irreducible lens contains at least two triangles incident to the boundary of the lens; unless the lens is L_0, they have no edge in common.

(3) If an irreducible lens $L \neq L_0$ of $I(G)$ is given, and if an ω_0 or η_0 reduction is performed on a 3-valent node or on a triangle of G corresponding to a triangle of L having an edge in the boundary of L, the resulting graph G^* has the following property: There is a lens (and hence, by (1), an irreducible lens L^*) of $I(G^*)$ having fewer faces than L, *such that its faces correspond to some of the faces of* L.

Combining (1), (2) and (3) the proof of Steinitz's theorem is completed by starting from any region bounded by one or two geodesic arcs, finding an irreducible lens L in it, and applying a suitable ω_0 or η_0 reduction. Since L contains only finitely many faces, by repeated application of (3) we necessarily reach a graph G containing a lens of type L_0. But the presence of such a lens in $I(G)$ means that G contains a triangle with a trivalent node; hence to G the Case 1 is applicable. (In the proof of Steinitz's theorem L is chosen to contain the least number of faces among all lenses of $I(G)$, hence there is no necessity to use the italicized part of (3); however, this part of (3) is evident from the proof of the first part [1, p. 242].)

In view of the above, in order to complete the proof of our theorem it is sufficient to exhibit, for each G which is not covered by Case 1, a region R bounded by one or two geodesic arcs of I(G) and such that at most one of the triangles of R corresponds to a node of G belonging to F. Indeed, this property will be inherited by any irreducible lens L contained in R, and hence L will contain at least one other triangle such that the 3-valent node, or triangle, of G corresponding to it is disjoint from F. Thus an appropriate reduction ω_0 or η_0 will be applicable without interference with F, and F' may be chosen as the preassigned face of a 3-polytope realizing the reduced

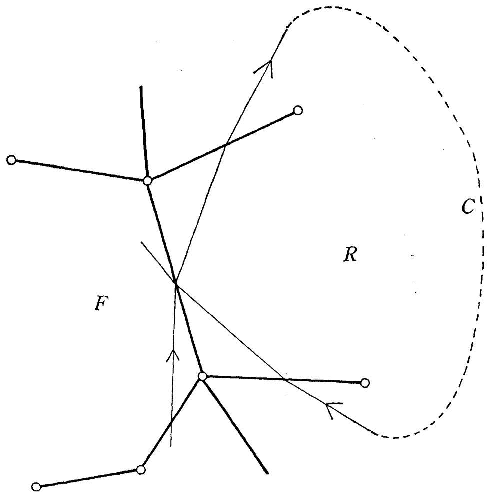

FIGURE 5

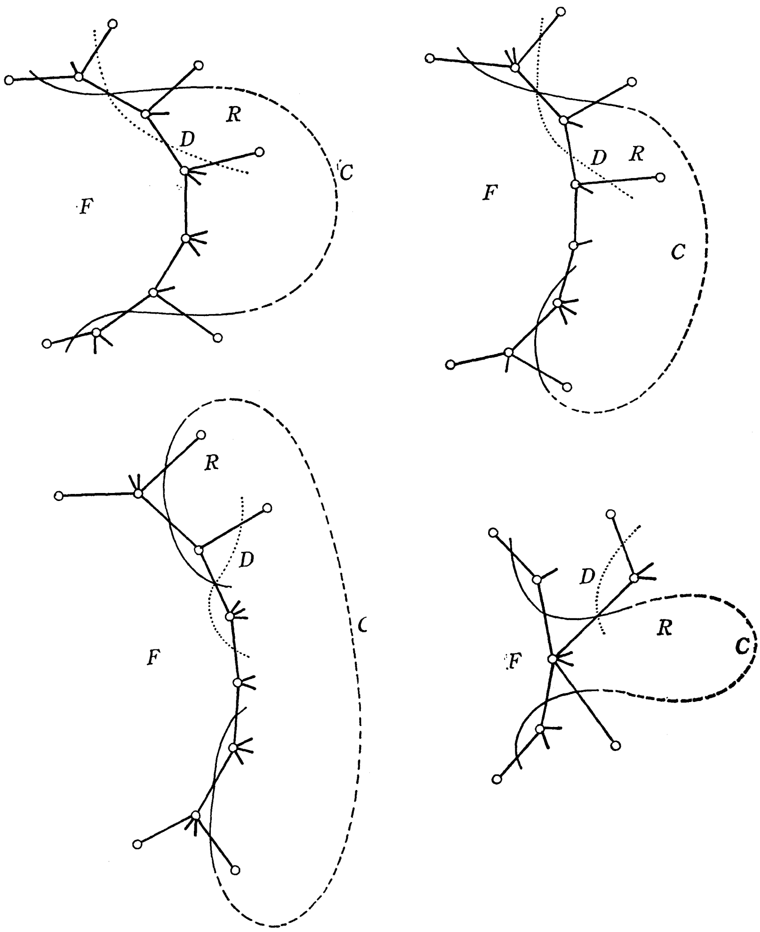

FIGURE 6

graph.

But the existence of a region R of the type required is easy to establish. Proceeding on any geodesic arc (starting with an edge of $I(G)$ having both endpoints on edges of F) we note that if C intersects itself prior to its return to F, or if it self-intersects while reentering F, that region enclosed by C which does not contain F may be taken as R. (Compare the schematic drawing in Figure 5.) Hence we are left only with the case in which each geodesic arc is free of self-intersections outside F. Among all the (simple) geodesic arcs of $I(G)$ having endpoints at edges of F, we choose one bridging the smallest possible number of edges of F and denote it by C. The possible situations are schematically indicated in Figure 6. In the first three cases the minimality of C prevents D from returning to F before meeting C again, and thus parts of C and D will determine a region R of the type required. In the fourth case D is any geodesic arc crossing C at a relatively interior point, and D either meets C before meeting F, or it meets C and F at the same vertex, again producing a region of the type required.

This completes the proof in Case 2, and with it the proof of the theorem.

REMARK 1. *As an easy corollary of the theorem we have:*
If P is a 3-polytope and if C is a simple closed circuit of edges of P such that no facet of P meets two edges of C, there exists a polytope P' combinatorially equivalent to P such that the circuit of P' corresponding to C is in a plane.

2. By an obvious application of duality, if follows from the theorem that the shape of one vertex-figure of a 3-polytope may be arbitrarily prescribed. Probably, more elements of a 3-polytope may be arbitrarily prescribed; however, it is easy to see that it is not always possible to preassign the shape of two faces having a common edge. (For example, the two quadrilaterals of Figure 7 may not appear in any 3-sided prism.) It would be interesting to investigate the following

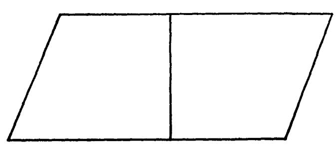

FIGURE 7

CONJECTURE. For any family $\{F_1, \cdots, F_n\}$ of disjoint faces of a 3-polytope P, and for any family $\{F_1', \cdots, F_n'\}$ of polygons, F_i' being of the same type as F_i, there exists a 3-polytope P' combinatorially equivalent to P, such that every face of P' corresponding to one of the faces F_i is projectively equivalent to F_i'.

3. It would be very interesting to determine to what extent the theorem holds in higher dimensions. The only known result in this direction seems to be M.A. Perles' example of an 8-dimensional polytope P with 12 vertices such that the shape of one of its 7-dimensional faces (with 10 vertices) may not be arbitrarily chosen within its combinatorial type ([1, p. 96, Exercise 3]). It may be conjectured that a similar failure of the theorem occurs already in four dimensions.

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