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JORDAN ALGEBRAS AND EXCEPTIONAL SUBALGEBRAS OF THE EXCEPTIONAL ALGEBRA E_6

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The close relationship which exists between exceptional central simple Lie algebras, Cayley algebras, and exceptional central simple Jordan algebras has been known for some time. The representational point of view which the latter nonassociative algebras afford has led to the complete classification of the Lie algebras G_2 and F_4, partial classification of the Lie algebras D_4 and E_6, and to concrete realizations for forms of the above algebras and the algebras E_7 and E_8.

In the present paper we shall establish a "coordinatization" theorem (Theorem 2) for exceptional simple subalgebras of the Lie algebra $\mathfrak{L}(\mathfrak{J})$ of type E_6, over an algebraically closed field of characteristic 0, in terms of the annihilated subspace. We use this to give a new proof of the well known conjugacy (see Dynkins Table 25) of split subalgebras of type G_2 or D_4 or F_4, of a split algebas of type D_4 or F_4 or E_6 over a field of characteristic 0 (Theorem 3). This is then applied to obtain new results in the classification of D_4 and E_6 which are subsequently used in generalizing the above conjugacy and extension of automorphism theorems to the (possibly) nonsplit case.

Throughout this paper, unless specifically stated otherwise, all fields which appear will have characteristic 0. If $\mathfrak{L}$ is a Lie algebra over the field k, then we say that $\mathfrak{L}$ is (a form) of type X_l if $\mathfrak{L}_{\bar{k}}$ ($\bar{k}$ the algebraic closure of k) is the Lie algebra X_l in the Killing-Cartan-Seligman classification.

1.2. Let $\mathfrak{C}$ be a Cayley algebra over the field k. Recall that $\mathfrak{C}$ is an 8-dimensional vector space together with a nondegenerate bilinear form $n(a, b)$, and a bilinear multiplication $\mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$ ($(a, b) \mapsto ab$) which are related by

$$n(ab, ab) = n(a, a)n(b, b) \qquad (a, b) \in \mathfrak{C} .$$

$\mathfrak{C}$ is a unital, central simple, alternative, notassociative algebra and n necessarily has Witt index 0 or 4 ([14]). In the latter case $\mathfrak{C}$ is referred to as the split Cayley algebra over k. It is well known that $\mathfrak{D}(\mathfrak{C})$, the derivation algebra of a split Cayley algebra, is the split Lie algebra G_2 ([12]) and that the Lie algebra $\mathfrak{s}(\mathfrak{C}, n)$ of n-skew transformations in $\mathfrak{C}$ is the split Lie algebra D_4 ([15]).

THEOREM 1. *Let $\mathfrak{C}$ be a Cayley algebra over a field k (char k =*

0), $\mathfrak{D} = \mathfrak{D}(\mathfrak{C})$ *and* $\mathfrak{L} = \mathfrak{s}(\mathfrak{C}, n)$. *Then every isomorphism* $\alpha : \mathfrak{D} \to \mathfrak{L}$ *is extendable to an automorphism of* $\mathfrak{L}$. *If* k *is algebraically closed then extension can be achieved by an invariant automorphism* ([15] p. 265) *of* $\mathfrak{L}$.

Proof. It is known (c.f. [15], p. 234) that the only irreducible module for the split G_2, of dimension at most eight, is seven dimensional and is unique. This implies that $\mathfrak{D}(\mathfrak{C})$ has a unique 7-dimensional module.

$\mathfrak{C}$ is a completely reducible $\mathfrak{K} \equiv \mathfrak{D}^{\alpha}$-module (char $k = 0$, c.f. [15], p. 79) and the above shows that $\mathfrak{C} = \mathfrak{Z} \oplus \mathfrak{M}$ where $\mathfrak{Z}$ is a 1-dimensional zero module and $\mathfrak{M}$ is an irreducible 7-dimensional $\mathfrak{K}$-module. One easily sees that $\mathfrak{Z}$ and $\mathfrak{M}$ are the only nontrivial $\mathfrak{K}$-submodules of $\mathfrak{C}$. Since $\mathfrak{K}$ consists of n-skew transformations, $\mathfrak{Z}^{\perp}$ is a 7-dimensional $\mathfrak{K}$-submodule so $\mathfrak{Z}^{\perp} = \mathfrak{M}$. The corresponding decomposition of $\mathfrak{C}$ as $\mathfrak{D}$-module is $C = k1 + \mathfrak{C}_0$ where $\mathfrak{C}_0$ is the (-1)-space of the canonical involution $a \to n(a, 1)1 - a \equiv \bar{a}$ in $\mathfrak{C}$.

If we identify $\mathfrak{D}$ and $\mathfrak{K}$ by α, then the uniqueness of the 7-dimentional $\mathfrak{D}(\mathfrak{C})$-module manifests itself by the existence of a linear isomorphism $B\colon \mathfrak{C}_0 \to \mathfrak{M}$ such that

$$(1) \qquad (c_0 d)B = (c_0 B)d^{\alpha} \quad \text{for all} \quad c_0 \in \mathfrak{C}_0,\, d \in \mathfrak{D}\,.$$

Define a nondegenerate bilinear form n' on $\mathfrak{M}$ by $n'(m, m') = n(mB^{-1}, m'B^{-1})$. A simple calculation shows that $\mathfrak{K}$ is skew with respect to n'. Since $\mathfrak{K}$ generates $\mathrm{End}_k\, \mathfrak{M}$ (the representation is absolutely irreducible) and is skew with respect to both n' and $n \mid \mathfrak{M}$ it follows that the adjoints with respect to the two forms are the same and hence that the forms are dependent (e.g. [15], p. 312, or [3]). Thus there is a $\lambda \in k^*$ with $n' = \lambda n \mid \mathfrak{M}$. For $m \in \mathfrak{M}$, $\lambda n(m, m) = n'(m, m) = n(mB^{-1}, mB^{-1})$, so there is an $a \in \mathfrak{C}$ with $n(a, a) = \lambda$. $Ba_R (a_R\colon b \to ba)$ is an orthogonal mapping of $\mathfrak{C}_0$ into $\mathfrak{C}$ and by Witt's Theorem there exists an $0 \in O(n)$ with $Ba_R = 0 \mid \mathfrak{C}_0$. Let $A = 0a_R^{-1}$. A is a similitude of $(\mathfrak{C}, n)$ and conjugation by A is an automorphism of $\mathfrak{L}$. Since $A \mid \mathfrak{C}_0 = B$ and $(k1)A = \mathfrak{Z}$, (1) shows that this automorphism extends α.

1.3. In this section we introduce the exceptional central simple Jordan algebra, recall some well known results for further use, and indicate the canonical realizations of the algebras G_2, D_4, F_4 and E_6 in terms of these algebras.

Let $\mathfrak{C}$ be a Cayley algebra over the field k and consider the algebra $\mathfrak{C}_3 = \mathfrak{C} \otimes_k k_3$ of all 3×3 matrices with entries in $\mathfrak{C}$. If $\gamma_i \in k^*$, $i = 1, 2, 3$, then the subspace $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$ of all matrices in $\mathfrak{C}_3$ of the form

$$\begin{pmatrix} \alpha_1 & a_3 & \gamma_1^{-1}\gamma_3\bar{a}_2 \\ \gamma_2^{-1}\gamma_1\bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \gamma_3^{-1}\gamma_2\bar{a}_1 & \alpha_3 \end{pmatrix} \alpha_i \in k,\ a_i \in \mathfrak{C}$$

(and $a \to \bar{a}$ the canonical involution in $\mathfrak{C}$) is equipped with the structure of an exceptional central simple Jordan algebra by means of the composition $x.y = (1/2)(xy + yx)$, xy denoting the product in $\mathfrak{C}_3$.

If we let $\{e_{ij}\}$ be the usual matrix units in $k_3 \subseteqq \mathfrak{C}_3$ then $e_{ii} = e_i$ are orthogonal idempotents in $\mathfrak{J}$, $i = 1, 2, 3$, and $I = e_1 + e_2 + e_3$ is the identity of $\mathfrak{J}$.

$\mathfrak{J}$ is a power associative algebra and the generic minimal polynomial of $x \in \mathfrak{J}$ is

$$(2) \qquad x^{\cdot 3} - T(x)x^{\cdot 2} + Q(x)x - N(x)I = 0$$

where $T(x)$ is the (linear) generic trace form, $Q(x)$ a quadratic form, and $N(x)$ the (cubic) generic norm form. The trace bilinear form $T(x, y) = T(x \cdot y)$ is symmetric and nondegenerate and if we let $N(x, y, z)$ be the linearized norm form then $T(x) = 3N(x, I, I)$ ([16] III, p. 69, eq. 25), and we can introduce the Freudenthal cross product $x \times y$ by requiring that

$$T(x \times y, z) = 3N(x, y, z) \quad \text{for all} \quad z \in \mathfrak{J}\,.$$

One can obtain $x \times y$ explicitly from the multiplication in $\mathfrak{J}$ as

$$x \times y = x \cdot y - \frac{1}{2}T(x)y - \frac{1}{2}T(y)x + \frac{1}{2}(T(x)T(y) - T(x \cdot y))I$$

([11], eq. 1.4). Using this form of the cross product we see that

$$(0\colon e_1) \equiv \{x \in \mathfrak{J} \mid x \times e_1 = 0\} = ke_1 \oplus \mathfrak{J}_{12} \oplus \mathfrak{J}_{13}$$

(where $\mathfrak{J}_{ij} = \{a_{ij} = ae_{ij} + \gamma_j^{-1}\gamma_i\bar{a}e_{ji} \mid a \in \mathfrak{C}\} \subseteqq \mathfrak{J}$) and hence that $\dim(0\colon e_1) = 17$.

The Peirce decomposition of $\mathfrak{J}$ relative to $\{e_i\}$ is $\sum_{i=1}^{3} ke_i + \sum_{i<j}\mathfrak{J}_{ij}$ and is an orthogonal decomposition with respect to the trace bilinear form. It then follows that $(0\colon e_1)^{\perp} \equiv \mathfrak{R} = ke_2 + ke_3 + \mathfrak{J}_{23}$ is a Jordan subalgebra of $\mathfrak{J}$ with identity $e_2 + e_3$. We note that for $x \in \mathfrak{R}$, $T(x, e_2 + e_3) = 0$ implies $x^{\cdot 2} \in k(e_2 + e_3)$.

Assume for the moment that $\mathfrak{J}$ is an arbitrary exceptional central simple Jordan algebra over k (i.e., that $\mathfrak{J}_{\bar{k}}$ is an algebra of the preceding type). $\mathfrak{J}$ is called reduced if it contains a nontrivial idempotent and one has the result of Schafer ([20], [22]) that every reduced algebra has the form $\mathfrak{H}(\mathfrak{C}_3, \gamma)$ where the Cayley algebra is unique up to isomorphism. Following Jacobson ([13]) we introduce the ternary composition $\{xyz\} = (x \cdot y) \cdot z + (y \cdot z) \cdot x - (z \cdot x) \cdot y$ in $\mathfrak{J}$. It is known that if u is an element of $\mathfrak{J}$ with $N(u) \neq 0$ (hence u is invertible) then the composition $(x, y) \mapsto \{xuy\} \equiv x \cdot_u y$ equips the underlying vector space of $\mathfrak{J}$ with the structure of an exceptional central simple Jordan algebra which is denoted by $\mathfrak{J}^{(u)}$ and which is called the u-isotope of $\mathfrak{J}$. The identity of $\mathfrak{J}^{(u)}$ is $v = u^{-1}$, $\mathfrak{J}^{(u)}$ is reduced if and only if $\mathfrak{J}$ is reduced and if this is the case, then the coordinatizing Cayley algebras are isomorphic ([1]). If $N^{(u)}(x)$ is the generic norm on $\mathfrak{J}^{(u)}$, then

$$(3) \qquad N^{(u)}(x) = N(x)N(u) \qquad ([17]) .$$

Finally $\mathfrak{J}$ is called split if it is reduced and if the attached Cayley algebra is split.

Let $\Pi \equiv \Pi(\mathfrak{J}) \equiv \{x \in \mathfrak{J} \mid x \neq 0, x \times x = 0\} = \{x \in \mathfrak{J} \mid x \neq 0, N(x, x, y) = 0.$ for all $y\}$ be the elements of rank one in $\mathfrak{J}$. It is known that Π consists exactly of all nonzero elements in $\mathfrak{J}$ which are either nilpotent of order 2 $(x \in \Pi,\ T(x) = 0)$ or scalar multiples of primitive idempotents $(x \in \Pi,\ T(x) \neq 0)$. The conditions that $x \in \mathfrak{J}$ be a primitive idempotent are $x \in \Pi$ and $T(x) = 1$ ([22]). Using (3) we see that $(\Pi\mathfrak{J}) = \Pi(\mathfrak{J}^{(u)})$ for every invertible element u of $\mathfrak{J}$. It is known that $\Pi(\mathfrak{J})$ spans $\mathfrak{J}$.

PROPOSITION 1. *Let $x \in \Pi(\mathfrak{J})$, $\mathfrak{J}$ split. Then there is an isotope $\mathfrak{J}^{(u)}$ of $\mathfrak{J}$ that such x is a primitive idempotent in $\mathfrak{J}^{(u)}$.*

Proof. If $N(x, y, z) = 0$ for all $y, z \in \Pi$ then $N(x, y, z) = 0$ for all $y, z \in \mathfrak{J}$ and hence $x \times y = 0$ for all $y \in \mathfrak{J}$. In particular $0 = x \times I = (1/2)(T(x)I - x)$ which is absurd. Thus there exist $y, z \in \Pi$ with $N(x, y, z) \neq 0$. It follows that $v = x + y + z$ is invertible and that $N(v) = 6N(x, y, z)$. In the $u = v^{-1}$ isotope of $\mathfrak{J}$, $x \in \Pi(\mathfrak{J}^{(u)})$ and $T^{(u)}(x) = 3N^{(u)}(x, v, v) = 6N(u)N(x, y, z) = 1$, so x is a primitive idempotent in $\mathfrak{J}^{(u)}$.

Let $\mathfrak{J}$ be a split exceptional central simple Jordan algebra over k and $\{e_i\}$ a set of three supplementary orthogonal idempotents. $\mathfrak{L}(\mathfrak{J}) \equiv \{L \in \mathrm{End}_k \mathfrak{J} \mid N(xL, x, x) = 0$ for all $x \in \mathfrak{J}\}$— the algebra of norm skew transformations in $\mathfrak{J}$— is the split Lie algebra E_6([7]). $\mathfrak{D}(\mathfrak{J}) \equiv \{$derivations of $\mathfrak{J}\} = \{D \in \mathfrak{L}(\mathfrak{J}) \mid ID = 0\} = \{D \in \mathfrak{L}(\mathfrak{J}) \mid -D^\sharp = D\}$, $\sharp$ denoting transpose with respect to the trace form, is the split Lie algebra F_4 ([7]). $\mathfrak{D}(\mathfrak{J}/\Sigma k e_i) \equiv \{D \in \mathfrak{D}(\mathfrak{J})$ or $\mathfrak{L}(\mathfrak{J}) \mid e_i D = 0,\ i = 1, 2, 3\}$ is the split Lie algebra D_4 ([7]). If $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$ the subspace $\mathfrak{h} = \Sigma k e_j + \Sigma_{i<j} k 1_{ij}$ is a simple subalgebra of $\mathfrak{J}$. Indeed $\mathfrak{g} = \mathfrak{h}(k_3, \gamma)$— the symmetric 3×3 matrices over k relative to the involution $(x_{ij}) \to \gamma^{-1}(x_{ji})\gamma$, where $\gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\}$. $\mathfrak{h}$ is isomorhic also to the algebra of symmetric linear transformations in a three dimensional space relative to the quadratic form $\sum_{i=1}^3 \gamma_i X_i^2$. It is easy to see that $\mathfrak{D}(\mathfrak{J}/\mathfrak{h})$ is the split Lie algebra G_2.[1]

[1] The isomorphism d_{23} described in [2], p. 251, carries $\mathfrak{D}(\mathfrak{J}/\mathfrak{h})$ onto $\mathfrak{D}(\mathfrak{C})$.

PROPOSITION 2. *Let $\mathfrak{J}$ be a split exceptional central simple Jordan algebra over k and let $GL(\mathfrak{J})$ be the group of all norm equivalences of $\mathfrak{J}(N(xT) = \lambda N(x)$ for all $x \in \mathfrak{J}, \lambda$ fixed). Then $GL(\mathfrak{J})$ acts transitively on the set of invertible elements of $\mathfrak{J}(N(x) \neq 0)$ and on $\Pi(\mathfrak{J})$.*

Proof. The last part is a consequence of Proposition 1 and a result of Jacobson's ([16] II, Th. 5). If $v \in \mathfrak{J}$ with $N(v) \neq 0$ then the algebras $\mathfrak{J}$ and $\mathfrak{J}^{(u)}$, $(u = v^{-1})$ are isomorphic. If T is an isomorphism between them then T is an equivalence between their norm forms and the conclusion follows immediately from (3) and the fact that $IT = v$, the identity of $\mathfrak{J}^{(u)}$.

1.4 Canonical embeddings. Let $\mathfrak{J}$ be a split exceptional central simple Jordan algebra over k, k algebraically closed of characteristic $\neq 2, 3$. Throughout this section $\mathfrak{K}$ will be a Lie subalgebra of $\mathfrak{L}(\mathfrak{J})$ which acts completely reducibly on $\mathfrak{J}$. For any $\mathfrak{K}$-module $\mathfrak{M}$ we let $\mathfrak{M}_0 \equiv \mathfrak{M}_0(\mathfrak{K})$ be the submodule of $\mathfrak{M}$ annihilated by $\mathfrak{K}$. For convenience we shall call $\mathfrak{M}$ nondegenerate if $\mathfrak{M}_0 = \{0\}$. It is easy to see that if M is a subspace of a nondegenerate completely reducible $\mathfrak{K}$-module, then the submodule generated by M is spanned (as a vector space) by $\{mL \mid m \in M, L \in \mathfrak{K}\}$ over k.

LEMMA 1. *Suppose that $\mathfrak{J}_0(\mathfrak{K}) \equiv \mathfrak{J}_0 \neq \{0\}$ and that $\mathfrak{J}$ contains no 10-dimensional nondegenerate $\mathfrak{K}$-submodule. Then there is a $u \in \mathfrak{J}_0$ where $N(u) \neq 0$.*

Proof. Suppose that $N(x) = 0$ for all $x \in \mathfrak{J}_0$. Since $\mathfrak{J}$ is a completely reducible $\mathfrak{K}$-module, $\mathfrak{J} = \mathfrak{J}_0 \oplus \mathfrak{M}$ where $\mathfrak{M}$ is a (nondegenerate) $\mathfrak{K}$-module. For $z \in \mathfrak{M}$ there are $w_i \in \mathfrak{M}$ and $L_i \in \mathfrak{K}$ such that $\sum w_i L_i = z$. Using the "norm skewness" of L we see that

$$N(x, y, z) = \sum (N(x, y, w_i L_i) + N(x, yL_i, w_i) + N(xL_i, y, w_i)) = 0$$

for all $x, y \in \mathfrak{J}_0$. Since $N(x, y, z) = 0$ for all $x, y, z \in \mathfrak{J}_0$ it follows that $x \times y = 0$ for all $x, y \in \mathfrak{J}_0$. $\mathfrak{L}(\mathfrak{J}) = \mathfrak{L}(\mathfrak{J}^{(u)})$ for all isotopes of $\mathfrak{J}$ by (3) so we may use this together with Proposition 1 to reduce the argument to the case where $\mathfrak{J}_0$ contains a primitive idempotent, say e. Since primitive idempotents are conjugate in $\mathfrak{J}$ ([1]) we may assume that $\mathfrak{J} = \mathfrak{H}(\mathfrak{C}_3, \gamma)$ where $e = e_1$ (notation as before). $(0{:}\, e)$ is *a* $\mathfrak{K}$-submodule of $\mathfrak{J}$ and $e \times \mathfrak{J}_0 = 0$ implies that $\mathfrak{J}_0 \subseteqq (0{:}\, e)$. By complete reducibility $(0{:}\, e)$ has a 10-dimensional complement which is nondegenerate since $\mathfrak{J}_0 \subseteqq (0{:}\, e)$. This contradicts our assumption on $\mathfrak{J}$.

LEMMA 2. *Suppose that $\mathfrak{J}_0(\mathfrak{K}) \equiv \mathfrak{J}_0 \supsetneqq kI$, $n = \dim \mathfrak{J}_0$, and that $\mathfrak{J}$*

admits no $(n-1)$*-dimensional nondegenerate* $\mathfrak{K}$*-submodule. Then* $\mathfrak{J}_0$ *contains a primitive idempotent.*

Proof. Our hypotheses imply that $\mathfrak{K} \subseteqq \mathfrak{D}(\mathfrak{J})$. It suffices to show that $\mathfrak{J}_0$ contains a nontrivial idempotent e, since then either e or $I - e \in \mathfrak{J}_0$ will be primitive. Proceeding by contradiction we assume that I is the only idempotent in the subalgebra $\mathfrak{J}_0$.

For $x \in \mathfrak{J}_0$, let $k[x]$ be the (commutative, associative) subalgebra of $\mathfrak{J}$ generated by x and I. Since I is the only idempotent in $k[x]$, Wedderburn's Principal Theorem ([8], p. 491) shows that $k[x] = kI + N_x$ where N_x is the radical of $k[x]$ (k is algebraically closed). Thus $\mathfrak{J}_0$ is an almost nil Jordan algebra and hence $\mathfrak{J}_0 = kI + \mathfrak{N}$ where $\mathfrak{N}$ is a nilpotent ideal in $\mathfrak{J}_0$ ([19]). For $x \in N$, (2) reduces to $x^{\cdot 3} = 0$ ([17], Th. 1) so $0 = T(x) = T(x, I)$ and $\mathfrak{N} \subseteqq (kI)^{\perp}$, the orthogonal complement of kI relative to the trace bilinear form.

Since $\mathfrak{N}$ is a subalgebra of $\mathfrak{J}_0$, this computation also shows that $\mathfrak{N}$ is a totally isotropic subspace of $(kI)^{\perp}$. Thus $\mathfrak{N}^{\perp}$ is a $27 - (n-1)$-dimensional $\mathfrak{K}$-module which contains $\mathfrak{J}_0$. By complete reducibility it has a (nondegenerate) $(n-1)$-dimensional complement. This contradiction establishes the result.

LEMMA 3. *Suppose that* $\mathfrak{J}_0 = \mathfrak{J}_0(\mathfrak{K})$ *contains* I *and a primitive idempotent* e, *and let* $\mathfrak{R} = (0{:}\, e)^{\perp}$, $m = \dim(\mathfrak{J}_0 \cap \mathfrak{R})$ *and assume* $m \geqq 2$. *If* $\mathfrak{R}$ *contains no nondegenerate* $\mathfrak{K}$*-submodules of dimension* $m-1$, *then* $\mathfrak{J}_0$ *contains three supplementary, orthogonal idempotents.*

Proof. Without loss of generality we may take $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$ where $e = e_1$. By the discussion of §1.3 $\mathfrak{R} = ke_2 + ke_3 + \mathfrak{J}_{23}$. As in the proof of Lemma 2 ($\mathfrak{K} \subseteqq \mathfrak{D}(\mathfrak{J})$) we see that both $\mathfrak{R}$ and $\mathfrak{R}' = \mathfrak{R} \cap (k(e_2 + e_3)^{\perp})$ are $\mathfrak{K}$-submodules of $\mathfrak{J}$, and that if the desired conclusion does not hold then $\mathfrak{J}_0 \cap \mathfrak{R}$ is an almost nil Jordan algebra with nilradical $\mathfrak{N}$. $\mathfrak{N} \subseteqq \mathfrak{R}'$ and is totally isotropic since it consists of nilpotents of order two (c.f. remark in §1.3 following the introduction of the cross product). Then $\mathfrak{N}^{\perp} \cap \mathfrak{R}$ is a $10 - (m-1)$-dimensional submodule of $\mathfrak{R}$ containing $\mathfrak{R}_0(\mathfrak{K}) = \mathfrak{J}_0 \cap \mathfrak{R}$. Since $\mathfrak{R}$ is completely reducible $\mathfrak{K}$-module, a $\mathfrak{K}$-complement for $\mathfrak{N}^{\perp} \cap \mathfrak{R}$ in $\mathfrak{R}$ is a nondegenerate $(m-1)$-dimensional $\mathfrak{K}$-module in $\mathfrak{R}$. Contradiction.

We now have the main result of this section.

THEOREM 2. *Let* k *be an algebraically closed field of characteristic* 0, $\mathfrak{J}$ *an exceptional simple Jordan algebra over* k, *and* $\mathfrak{K}$ *a subalgebra of* $\mathfrak{L} = \mathfrak{L}(\mathfrak{J})$.

(a) *If $\mathfrak{K}$ is of type F_4, then there is a $u \in \mathfrak{J}_0(\mathfrak{K})$ with $N(u) \neq 0$ and $\mathfrak{K} = \mathfrak{D}(\mathfrak{J}^{(u)})$.*

(b) *If $\mathfrak{K}$ is of type D_4, then there is a $u \in \mathfrak{J}_0(\mathfrak{K})$ with $N(u) \neq 0$ and three supplementary orthogonal idempotents $\{e_i\}$ in $\mathfrak{J}_0^{(u)}(\mathfrak{K})$ with $\mathfrak{K} = \mathfrak{D}(\mathfrak{J}^{(u)}/\sum ke_i)$.*

(c) *If $\mathfrak{K}$ is of type G_2 and $\mathfrak{J}_0(\mathfrak{K}) \neq \{0\}$, then there are elements $u, \{e_i\} \in \mathfrak{J}_0(\mathfrak{K})$ as in* (b) *with $\mathfrak{h}(k_3) \cong \mathfrak{J}_0(\mathfrak{K}) \subseteqq \mathfrak{J}^{(u)}$ and $\mathfrak{K} = \mathfrak{D}(\mathfrak{J}^{(u)}/\mathfrak{h}(k_3))$.*

Proof. Since every module for a semisimple Lie algebra over an algebraically closed field of characteristic zero is completely reducible, $\mathfrak{K}$ satisfies the initial condition in the preceding lemmas. Let $\mathfrak{M}$ be a nondegenerate $\mathfrak{K}$-module with $d = \dim \mathfrak{M} \leqq 27$. Using the Weyl dimension formula ([15], p. 257) we obtain the following possibilities: F_4, $d = 26$; D_4, $d = 8$, 16 or 24; G_2, $d = 7, 14$ or 21.

(a) By Lemma 1 and the equivalent defining relations for $\mathfrak{D}(\mathfrak{J})$ we see that $\mathfrak{K} \subseteqq \mathfrak{D}(\mathfrak{J}^{(u)})$ for some u. Since the latter algebra is of type F_4 we obtain our first conclusion.

(b) By Lemma 1 and the above we see that $\mathfrak{K} \subseteqq \mathfrak{D}(\mathfrak{J}^{(u)})$ for some u. If $n = \dim \mathfrak{J}_0^{(u)}(\mathfrak{K})$ then $n = 19, 11$, or 3 and there are no nondegenerate submodules of dimension 18, 10, or 2. Lemma 2 shows that $\mathfrak{J}_0^{(u)}$ contains a primitive idempotent. In a similar way we see that the hypotheses of Lemma 3 are satisfied and thus obtain the second assertion since $\mathfrak{D}(\mathfrak{J}^{(u)}/\sum ke_i)$ is of type D_4.

(c) As in case (b) it follows that there exist $u, \{e_i\} \in \mathfrak{J}^{(u)}$ with $\mathfrak{K} \subseteqq \mathfrak{D}(\mathfrak{J}^{(u)}/\sum ke_i)$. Using Theorem 1, the canonical realization of the split G_2 in $\mathfrak{L}(\mathfrak{J})$ of §1.3 and the fact that every automorphism of $\mathfrak{D}(\mathfrak{J}^{(u)}/\sum ke_i)$ is realized as conjugation by an automorphism of $\mathfrak{J}$([2], p. 253) we see that $\mathfrak{J}_0(K) \cong \mathfrak{h}(k_3)$. This shows that $\mathfrak{K} = \mathfrak{D}(\mathfrak{J}^{(u)}/\mathfrak{h}(k_3))$ since the latter algebra is of type G_2.

Observe that the restriction in part (c) above is essential since $\mathfrak{L}(\mathfrak{J})$ contains an irreducible subalgebra of type G_2 ([9]).

COROLLARY. *Let k be an algebraically closed field of characteristic 0. Every subalgebra of type D_4 of an algebra of type E_6 is contained in a subalgebra of type F_4. A subalgebra of type G_2 of an algebra of type E_6 is in a subalgebra of type D_4 if and only if it is in a subalgebra of type F_4.*

1.5. Conjugacy theorems. Let $\mathfrak{C}$ be a Cayley algebra over k (char $k = 0$), $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$, $\{e_i\}$ the diagonal idempotents and $\mathfrak{h} = \mathfrak{h}(k_3, \gamma)$ the subalgebra $\sum ke_i \oplus \sum_{i<j} k1_{ij}$. Consider the following sequence

$$\mathfrak{D}(\mathfrak{J}/\mathfrak{h}) \subseteqq \mathfrak{D}(\mathfrak{J}/\sum ke_i) \subseteqq \mathfrak{D}(\mathfrak{J}) \subseteqq \mathfrak{L}(\mathfrak{J}) \, . \tag{4}$$

If k is algebraically closed then every automorphism of an algebra in the above chain extends to an (invariant) automorphism of any algebra of the sequence which contains it. If k is not algebraically closed then the above extendability still holds for the two sequences obtained by deleting either $\mathfrak{D}(\mathfrak{J}/\sum ke_i)$ or $\mathfrak{D}(\mathfrak{J})$ (even if $\mathfrak{J}$ is not split). One need only see this at each of the above inclusions. The first follows from (the translation of) Theorem 1, the second by [2] and the third by [21].

If $\mathfrak{K}$ is a subalgebra of type G_2 of a Lie algebra $\mathfrak{L}$ of type E_6 then we will call $\mathfrak{K}$ *o*-reducible (in $\mathfrak{L}$) if $\mathfrak{L}_0(\mathfrak{K}) \neq 0$ (relative to the representation $ad_{\mathfrak{L}} \mid \mathfrak{K}$). If $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{J})$ and if $\mathfrak{K}$ is a *o*-reducible G_2 subalgebra of $\mathfrak{L}$, then it is easy to see that $\mathfrak{J}_0(\mathfrak{K}) \neq \{0\}$. Indeed, one need only show this when k is algebraically closed. In this case, Weyl's demension formula shows that $\mathfrak{J}$ is either $\mathfrak{K}$-irreducible or $\mathfrak{J}_0(\mathfrak{K}) \neq \{0\}$. If $\mathfrak{J}$ is $\mathfrak{K}$-irreducible, then $\mathfrak{L}_0(\mathfrak{K}) \subsetneqq \{\text{centralizer } \mathfrak{K} \text{ in End } \mathfrak{J}\} = k$ (Shurs lemma). This implies that $\mathfrak{L}_0(\mathfrak{K}) = \{0\}$. Thus $\mathfrak{K}$ *o*-reducible in $\mathfrak{L}$ implies $\mathfrak{J}_0(\mathfrak{K}) \neq \{0\}$.

THEOREM 3. *Let $\mathfrak{K}_i$, $i = 1, 2$ be split subalgebras of type X_l of the split Lie algebra $\mathfrak{L}$ of type Y_n over a field k of characteristic 0. Then $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are conjugate in* $\mathrm{Aut}_k \mathfrak{L}$ *for the following choices of* (X_l, Y_n):

(a) (D_4, F_4) (d) (G_2, D_4)

(b) (D_4, E_6) (e) (*o*-reducible G_2, E_6) .

(c) (F_4, E_6)

Moreover, if k is algebraically closed, then we have also

(f) (G_2, F_4), *and in all cases the conjugacy can be obtained within* $(\mathrm{Aut}_k \mathfrak{L})_0$, *the group of invariant automorphisms of* $\mathfrak{L}$.

Proof. (b) Since $\mathfrak{L}$ is a split algebra of type E_6 we may take $\mathfrak{L} = \mathfrak{L}(\mathfrak{J})$, $\mathfrak{J}$ a split exceptional central simple Jordan algebra over k. By Theorem 2, $\mathfrak{J}_0(\mathfrak{K}_i)_{\bar{k}} = (\mathfrak{J}_{\bar{k}})_0(\mathfrak{K}_{i\bar{k}})$ is a diagonal algebra in a suitable isotope of $\mathfrak{J}_{\bar{k}}$. This implies that there is an invertible element u_i in $\mathfrak{J}_0(\mathfrak{K}_i)$ such that the subspace $\mathfrak{J}_0(\mathfrak{K}_i)$ is a cubic separable associative algebra in the u_i-isotope of $\mathfrak{J}$. Since $\mathfrak{K}_i = \mathfrak{D}(\mathfrak{J}^{(u_i)}/\mathfrak{J}_0(\mathfrak{K}_i))$ is split, it follows that $\mathfrak{J}_0^{(u_i)}(\mathfrak{K}_i)$ is a diagonal algebra ([2], Th. 5). Since an isotope of a split algebra is split, there exists an isomorphism $T: \mathfrak{J}^{(u_1)} \to \mathfrak{J}^{(u_2)}$ which carries $\mathfrak{J}_0(\mathfrak{K}_1)$ onto $\mathfrak{J}_0(\mathfrak{K}_2)$. As in the proof of Proposition 2, T is a norm equivalence of $\mathfrak{J}$ and conjugation by T induces an automorphism of $\mathfrak{L}$ which carries $\mathfrak{K}_1$ onto $\mathfrak{K}_2$.

(a) and (c) follow from obvious modifications of the above argument. (d) was established in Theorem 1.

(e) Again let $\mathfrak{L} = \mathfrak{L}(\mathfrak{J})$ as in (b). An argument analogous to that above shows that there are u_i such that $\mathfrak{J}_0^{(u_i)}(\mathfrak{K}_i)$ is a k-form of $\mathfrak{h}(\bar{k}_3)$.

Using the classification of forms of this algebra, we see that $\mathfrak{J}_0^{(u_i)}(\mathfrak{K}_i)$ is reduced and hence that $\mathfrak{J}_0(\mathfrak{K}_i) \supseteqq \Sigma k e_j^{(i)}$, a 3-dimensional diagonal algebra in $\mathfrak{J}^{(u_i)}$. This implies that $\mathfrak{K}_i \subset \mathfrak{D}(\mathfrak{J}^{(u_i)}/\Sigma k e_j^i)$. Using (b) and Theorem 1 together with the initial remarks in this section we see that $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are conjugate in $\mathrm{Aut}_k \mathfrak{J}$. (f) is similar to (e).

The last assertion follows from the determination of $(\mathrm{Aut}_k \mathfrak{L})_0$ given in [21].

COROLLARY 1. *Let k be a field of characteristic 0, $\mathfrak{K}$ a split subalgebra of type G_2 or D_4 or F_4 of a split algebra of type D_4 or F_4 or E_6 and let α be an isomorphism of $\mathfrak{K}$ into $\mathfrak{L}$. Assume that if $\mathfrak{L}$ is F_4 then k is algebraically closed and that if $\mathfrak{K}$ is G_2 and $\mathfrak{L}$ is E_6 then $\mathfrak{K}$ and $\mathfrak{K}^\alpha$ are o-reducible. Then α extends to* $\mathrm{Aut}_k \mathfrak{L}$.

Proof. Apply the theorem to $\mathfrak{K}$ and $\mathfrak{K}^\alpha$ and use the remarks at the beginning of the section.

COROLLARY 2. *Let k be a field of characteristic 0 and let $\mathfrak{K}_1 \subset \mathfrak{K}_2 \subset \mathfrak{K}_3 \subset \mathfrak{L}$ and $\mathfrak{K}_1' \subset \mathfrak{K}_2' \subset \mathfrak{K}_3' \subset \mathfrak{L}$ be two sequences of split algebras of types G_2, D_4, F_4 and E_6.*

(a) *If k is algebraically closed, then there is an invariant automorphism α of $\mathfrak{L}$ with $\mathfrak{K}_i^\alpha = \mathfrak{K}_i'$ for $i = 1, 2, 3$.*

(b) *Without restriction on k there is an automorphism α of $\mathfrak{L}$ with $\mathfrak{K}_i^\alpha = \mathfrak{K}_i'$ for $i = 1, 2$.*

2.1. Our main tool in applying the previous results to the classification of algebras of type D_4 and E_6 will be Galois descent for (nonassociative) algebras (see [15] Chap. X for details). If $\mathfrak{A}$ is an algebra over L, then an algebra $\widetilde{\mathfrak{A}}$ over $k \subseteqq L$ is called a k-form of $\mathfrak{A}$ if $\widetilde{\mathfrak{A}}_L \equiv \widetilde{\mathfrak{A}} \otimes_k L \cong \mathfrak{A}$. If L is a finite dimensional Galois extension of k and η is a homomorphism of $G = \mathrm{gal}\,(L/k)$ into $\mathrm{Aut}_k \mathfrak{A}(s \longrightarrow \eta(s))$ such that $\eta(s)$ is s-linear, then it is well known that $\mathfrak{A}^{\eta(G)} = \{x \in \mathfrak{A} \mid x^{\eta(s)} = x \text{ for all } s \in G\}$ is a k-form of $\mathfrak{A}$ and that every k-form arises this way (up to isomorphism). η is called the precocycle of G (in $\mathrm{Aut}_k \mathfrak{A}$) associated to (corresponding to, arising from, etc.) $\mathfrak{A}^{\eta(G)}$.

If $\mathfrak{C}$ is the split Cayley algebra over L, $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$, $\mathfrak{h} = \mathfrak{h}(k_3, \gamma)$ then one has the following exact sequences (and their linear counterparts):

$$(5) \qquad \{1\} \longrightarrow \mathrm{Aut}_L(\mathfrak{J}; \mathfrak{h}) \xrightarrow{\xi} \mathrm{Aut}_L \mathfrak{D}(\mathfrak{J}/\mathfrak{h}) \longrightarrow \{1\}$$

$$(6)\ [24] \qquad \{1\} \longrightarrow \mathrm{Aut}_k \mathfrak{J} \xrightarrow{\xi} \mathrm{Aut}_k \mathfrak{D}(\mathfrak{J}) \longrightarrow \{1\}$$

(7) [2]

$$\{1\} \longrightarrow L^* \times L^* \times L^* \longrightarrow \Gamma L_k(\mathfrak{J}/\Sigma L e_i) \xrightarrow{\xi} \mathrm{Aut}_k \mathfrak{D}(\mathfrak{J}/\Sigma L e_i) \longrightarrow \{1\}$$

(8) [21] $\{1\} \longrightarrow L^* \longrightarrow \Gamma L_k(\mathfrak{J}) \xrightarrow{\xi} \mathrm{Aut}_k \mathfrak{L}(\mathfrak{J})$

where in each case $C\xi$ denotes conjugation by C (in the indicated algebra), $\Gamma L_k(\mathfrak{J})$ is the k-semilinear analogue of $GL(\mathfrak{J})$, $\Gamma L_k(\mathfrak{J}/\Sigma Le_i)$ is the subgroup of $\Gamma L_k(\mathfrak{J})$ which leaves ΣLe_i stable, and $\mathrm{Aut}_L(\mathfrak{J}; \mathfrak{h})$ is the subgroup of $\mathrm{Aut}_L \mathfrak{J}$ which fixes $\mathfrak{h}$ pointwise. In (8), the range of ξ is a subgroup of index 2 and a convenient representative for the other coset is $\Theta: X \rightarrow -X^\#$, $\#$ as before. For $C \in \Gamma L_k(\mathfrak{J}/\Sigma Le_i)$ it is known that $e_i C \in Le_{ip(C)}$, $p(C) \in S_3$, the symmetric group on three letters. One obtains (5) from the facts that every automorphism of $\mathfrak{D}(\mathfrak{C})$ is realized as conjugation by an automorphism of $\mathfrak{C}$ and that $\mathrm{Aut}_k \mathfrak{C} \cong \mathrm{Aut}_k(\mathfrak{J}; \mathfrak{h})$ under the correspondence defined in [2] (p. 251).

If $\mathfrak{L}$ is of type D_4 (resp. E_6) and is split by L, then we take $\mathfrak{L}_L = \mathfrak{D}(\mathfrak{J}/\Sigma Le_i)$ (resp. $\mathfrak{L}(\mathfrak{J})$), $\mathfrak{J}$ as above, and consider the precocycle η associated with $\mathfrak{L}$. $\mathfrak{L}$ determines a homomorphism $p: G = \mathrm{gal}(L/k) \rightarrow S_3$ (resp. S_2) which is defined by $p(s) = p(C(s))$ where $\eta(s) = C(s)\xi$ (resp. $p(s) = 0$ when $\eta(s) \in \mathrm{Im}\,\xi$ and 1 otherwise). The integer $|p(G)|$ is called the D_4 (resp. E_6) type of L and is indicated by a Roman numeral subscript, e.g., D_{4III}. If H is the kernel of p and F is the fixed field of H, then $\mathfrak{L}_F$ is of type D_{4I} (resp. E_{6I}). Within a given algebraic closure of k, F is characterized as the minimal such extension and we call F the canonical D_{4I} (resp. E_{6I}) field extension of $\mathfrak{L}$ ([2], [10]).

$\mathfrak{L}$ is called a "Jordan D_4" if there is an exceptional central simple Jordan algebra $\tilde{\mathfrak{J}}$ over k, and a cubic separable associative subalgebra $\mathfrak{k} \subseteq \tilde{\mathfrak{J}}$, such that $\mathfrak{L} \cong \mathfrak{D}(\tilde{\mathfrak{J}}/\mathfrak{k})$. Until recently, such algebras furnished the only known examples of exceptional D_4's ([4]).

A precocycle of G in $\mathrm{Aut}_k \mathfrak{L}(\mathfrak{J})$ of the form $s \rightarrow A(s)\xi \circ \Theta^{p(s)}$, where $A: s \rightarrow A(s)$ is a precocycle of G in $\mathrm{Aut}_k \mathfrak{J}$ and p is a homomorphism of G onto $\{0, 1\}$ (the integers mod 2), defines a (twisted) form of $\mathfrak{L}(\mathfrak{J})$ of type E_{6II}. If $k(\sqrt{\lambda})$ is the fixed subfield of L corresponding to the kernel of p, then one often denotes this form by $\mathfrak{L}(\mathfrak{J}^{A(G)})_\lambda$. Finally if $\mathfrak{L}$ is a k-form of $\mathfrak{L}(\mathfrak{J})$ we let $\mathfrak{L}^*$ denote the k-subalgebra of $\mathrm{End}_L \mathfrak{J}$ generated by $\mathfrak{L}$. If $\mathfrak{L}$ is of type E_{6I}, then $\mathfrak{L}^*$ is a k-form of $\mathrm{End}_L(\mathfrak{J})$ of exponent 1 or 3.

The next section is devoted to a proof of

THEOREM 4. *Let k be a field of characteristic zero.*

I. (a) *A k-form $\mathfrak{L}$ of type E_6 contains a form of type F_4 if and only if $\mathfrak{L} = \mathfrak{L}(\mathfrak{J})$ or $\mathfrak{L} = \mathfrak{L}(\mathfrak{J})_\lambda$ for some exceptional central simple Jordan algebra $\mathfrak{J}$ over k (and quadratic extension $k(\sqrt{\lambda})$).*

(b) *If $\mathfrak{L}$ is a k-form of type E_{6I}, and $\mathfrak{L}$ contains a subalgebra of type D_4 or a o-reducible subalgebra of type G_2 then index $(\mathfrak{L}^*) = 1$*

or 3 (index here is the usual index for central simple associative algebras).

(c) *If* $\mathfrak{L}$ *is a k-form of type* E_{6I} *and contains a subalgebra of type* D_{4I} *or* D_{4II}, *then* $\mathfrak{L} = \mathfrak{L}(\mathfrak{J})$, $\mathfrak{J}$ *reduced.*

II. *If* $\mathfrak{L}$ *is a k-form of type* D_4 *then* $\mathfrak{L}$ *is a Jordan* D_4 *if either*

(a) $\mathfrak{L}$ *contains a subalgebra of type* G_2 *or*

(b) $\mathfrak{L}$ *is contained in an algebra of type* F_4 *or* E_{6I}.

We note that I(a) was originally obtained by R. B. Brown by different methods ([6]) and also that if there are no exceptional Jordan division algebras over k then the conclusion in I(b) can be sharpened to say that $\mathfrak{L}$ is obtainable by a Tits construction ([23]) (see [10] for details).

2.2. Throughout this section $\mathfrak{L}$ will be a form of type E_6 (resp. D_4) over k (char $k = 0$), $\mathfrak{K} \subsetneqq \mathfrak{L}$ a subalgebra of type D_4 or (o-reducible) G_2 (resp. G_2) and L a finite dimensional Galois extension of k with both $\mathfrak{L}_L$ and $\mathfrak{K}_L$ split. We take $\mathfrak{L}_L = \mathfrak{L}(\mathfrak{J})$ (resp. $\mathfrak{D}(\mathfrak{J}/\Sigma Le_i)$ where $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$ ($\mathfrak{C}$ the split Cayley algebra over L) and by Corollary 1 to Theorem 3 we may assume that $\mathfrak{K}_L$ is canonically embedded (as indicated in (4)). By (8) (resp. (7)) the precocycle of $G = \mathrm{gal}\,(L/k)$ in $\mathrm{Aut}_k \mathfrak{L}_L$ corresponding to $\mathfrak{L}$ is given by $\eta(s) = C(s)\xi \circ \Theta^{p(s)}$ (resp. $\eta(s) = C(s)\xi$) for $C(s) \in \Gamma L_k(\mathfrak{J})$ (resp. $\Gamma L_k(\mathfrak{J}/\Sigma Le_i)$) as indicated in the preceding section. Since Θ fixes $\mathfrak{D}(\mathfrak{J})$, $C(s)\xi$ leaves $\mathfrak{K}_L$ invariant and thus $\mathfrak{J}_0(\mathfrak{K}_L)$ is $C(s)$-stable for all s.

I(a). If $\mathfrak{L}$ is of type E_6, $\mathfrak{K}$ of type F_4, then this implies that $IC(s) = \Psi(s)I$, $\Psi(s) \in L^*$ and hence by [16] $A(s) = C(s)\Psi(s)^{-1}$ is an s-semiautomorphism of $\mathfrak{J}$ with $A(s)\xi \circ \Theta^{p(s)} = \eta(s)$. $A: s \rightarrow A(s)$ is easily seen to be a precocycle of G in $\mathrm{Aut}_k \mathfrak{J}$ and $\mathfrak{L}$ is either $\mathfrak{L}(\mathfrak{J}^{A(G)})$ or $\mathfrak{L}(\mathfrak{J}^{A(G)})_\lambda$ for suitable λ (where $\mathfrak{K} = \mathfrak{D}(\mathfrak{J}^{A(G)})$). The converse follows immediately from the realizations $\mathfrak{L}(\mathfrak{J}^{A(G)}) = R \oplus \mathfrak{D}(\mathfrak{J}^{A(G)})$, $\mathfrak{L}(\mathfrak{J}^{A(G)})_\lambda = \sqrt{\lambda}\, R + \mathfrak{D}(\mathfrak{J}^{A(G)})$ as k-subalgebras of $\mathfrak{L}(\mathfrak{J})$ where R is the k-space of right multiplications by elements of trace zero in $\mathfrak{J}^{A(G)}$.

I(b). If $\mathfrak{L}$ is of type E_{6I} then $\eta(s) = C(s)\xi$ where $C(s)C(h) = C(sh)\delta_{s,h}$, $\delta_{s,h} \in L^*$. $\mathfrak{L}^*$ is clearly the centralizer of the crossed product algebra $\mathfrak{X} \equiv (L, G, \delta) = \Sigma LC(s)$ in $\mathrm{End}_k \mathfrak{J}$. The centralizer of $\mathfrak{X} \mid \mathfrak{J}_0(\mathfrak{K}_L)$ in $\mathrm{End}_k \mathfrak{J}_0(\mathfrak{K}_L)$ is a k-form of $\mathrm{End}_L \mathfrak{J}_0(\mathfrak{K}_L)$ which has the same index as $\mathfrak{L}^*$. If $\mathfrak{K}$ is of type D_4 or ($\circ$-reducible) G_2, then consideration of the dimension of $\mathfrak{J}_0(\mathfrak{K}_L)$ and the relationship between index and exponent yields I(b).

I(c). As in the preceding case, $\eta(s) = C(s)\xi$ where $(\Sigma Le_i)C(s) = \Sigma Le_i$. Since $\mathfrak{R}$ is a special form of D_4 (i.e., of type D_{4I} or D_{4II}, see [18]), there is a j such that Le_j is $C(s)$-stable. This implies that we may assume $C(s)C(h) = C(sh)$ for all $s, h \in G$. As in [2], (Th. 6) there is a u-isotope ($u \in \Sigma Le_i$) of $\mathfrak{J}$ in which $C(s) \in \mathrm{Aut}_k \mathfrak{J}^{(u)}$. Since $\mathfrak{L}(\mathfrak{J}) = \mathfrak{L}(\mathfrak{J}^{(u)})$ we see that $\mathfrak{L} = \mathfrak{L}((\mathfrak{J}^{(u)})^{C(G)})$ where C is the precocycle $s \to C(s)$ of G in $\mathrm{Aut}_k \mathfrak{J}^{(u)}$. $(\mathfrak{J}^{(u)})^{C(G)}$ is reduced since it contains a scalar multiple of e_j.

Consider now the case where $\mathfrak{L}$ is of type D_4. It is known ([2], Th. I) that $\mathfrak{L}$ is a Jordan D_4 if and only if $\mathfrak{L}_F$ is a Jordan D_{4I} (where F is the canonical D_{4I} field extension of $\mathfrak{L}$), so we turn our attention to algebras of this type (since our hypotheses go up).

II(b). Since the idempotent spaces Le_i are $C(s)$-stable, $C(s)$ defines three s-linear transformations $C_k(s)$ in C via $a_{ij}C(s) = (aC_k(s))_{ij}$ ($a_{ij} \in \mathfrak{J}_{ij}$, $ij = 23, 31, 12$, i, j, k unequal). By the results of ([18], [2]) $C_k(s)C_k(h) = C_k(sh)\delta_k(s, h)$ where the δ_k are factor sets of order 2 whose product is split. $\mathfrak{L}$ is a Jordan D_{4I} if and only if each δ_k is itself split. If $\mathfrak{L}$ is contained in an algebra of type E_{6I}, then as above we see that $C(s)C(h) = C(sh)\delta(s, h)$ and hence that $\delta_k = \delta$, $k = 1, 2, 3$. The above remarks imply that δ is split and hence that $\mathfrak{L}$ is a Jordan D_4. Since every F_4 is contained in an E_{6I} this establishes II(b).

II(a). If $\mathfrak{K}$ is of type G_2 contained in $\mathfrak{L}$, then as in the proof of I(a) we see that I_{ij} is $C(s)$-stable and hence that each δ_k is split.

2.3. In this section we drop the assumption that the algebras in question are split to obtain more general results related to those in §1.5.

THEOREM 5. *Let $\mathfrak{J}$ be an exceptional central simple Jordan algebra over k* (char $k = 0$) *and suppose $\mathfrak{L}(\mathfrak{J})$ contains a $\circ$-reducible subalgebra $\mathfrak{K}$ of type G_2. Then $\mathfrak{J}$ is reduced and every isomorphism $\alpha: \mathfrak{K} \to \mathfrak{L} \equiv \mathfrak{L}(\mathfrak{J})$ such that $\mathfrak{K}^\alpha$ is $\circ$-reducible extends to an automorphism of $\mathfrak{L}$ of the form $X \to C^{-1}XC$, $C \in GL(\mathfrak{J})$.*

Proof. By the proof of Theorem 3(e), $\mathfrak{J}_0(K) \cong \mathfrak{h}(k_3, \gamma')$ (in a suitable isotope) and hence $\mathfrak{J}$ is reduced, say $\mathfrak{J} = h(\mathfrak{C}_3, \gamma)$. Since $\mathfrak{D}(\mathfrak{J}/\mathfrak{h}(k_3, \gamma))$ is a reducible G_2 in $\mathfrak{L}(\mathfrak{J})$ if suffices to consider only this case. The extendibility in (4) shows that every automorphism of $\mathfrak{D}(\mathfrak{J}/\mathfrak{h}(k_3, \gamma)) = \mathfrak{K}$ extends to an automorphism of $\mathfrak{L}(\mathfrak{J})$ of the desired type. To conclude the proof we need only show that if $\mathfrak{K}'$ is a $\circ$-reducible G_2-subalgebra of $\mathfrak{L}(\mathfrak{J})$ then there is an $\alpha' \in \mathrm{Aut}\,\mathfrak{L}(\mathfrak{J})$ ($X \mapsto X^{\alpha'} = C^{-1}XC$, $C \in GL(\mathfrak{J})$) with $\mathfrak{K}'^{\alpha'} = \mathfrak{D}(\mathfrak{J}/\mathfrak{h}(k_3, \gamma))$.

By our initial remarks, $\mathfrak{J}_0(\mathfrak{K}') \cong h(k_3, \delta)$ in some u-isotope of $\mathfrak{J}$. Since $\mathfrak{h}(k_3, \delta)$ is reduced, it contains a diagonal algebra Σkf_i, and thus $\mathfrak{K}' \subset \mathfrak{D}(\mathfrak{J}^{(u)}/\Sigma kf_i)$. By the 3-point transitivity of $GL(\mathfrak{J})$ on $\Pi(\mathfrak{J})$ (see [16], Prop. 13 and [1]), there is a $C' \in GL(\mathfrak{J})$ with $f_iC' \in ke_i$ ($\{e_i\}$ the diagonal idempotents in $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$). Conjugation by C' is an automorphism of $\mathfrak{L}(\mathfrak{J})$ and by our choice of C', $C'^{-1}\mathfrak{K}'C' \subset \mathfrak{D}(\mathfrak{J}/\Sigma ke_i)$. By Theorem 1 and the extendibility indicated in (4), there is a $C'' \in GL(J)$ with $\mathfrak{D}(\mathfrak{J}/h(k_3, \gamma)) = C''^{-1}C'^{-1}\mathfrak{K}'C'C''$.

COROLLARY 1. *Let k, $\mathfrak{K}$, and $\mathfrak{L}$ be as above. Then any automorphism of $\mathfrak{K}$ extends to an automorphism of $\mathfrak{L}$.*

COROLLARY 2. *Let k, $\mathfrak{K}$ and $\mathfrak{L}$ be as above. Then $\mathfrak{L}$ is split if and only if $\mathfrak{K}$ is.*

Proof. $\mathfrak{J}$ is necessarily reduced by Theorem 5, and the proof of that theorem together with Theorem 1 imply $\mathfrak{K} \cong \mathfrak{D}(\mathfrak{C})$ where $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$. The above conclusion follows from the fact that $\mathfrak{J}$ (hence $\mathfrak{L}$) is split if and only if $\mathfrak{C}$ (hence $\mathfrak{K}$) is split.

We note that if $\mathfrak{J}$ is a Jordan division algebra, $\mathfrak{k}$ a cubic subfield of $\mathfrak{J}$ then $\mathfrak{D}(\mathfrak{J})$ and $\mathfrak{D}(\mathfrak{J}/\mathfrak{k})$ provide examples of algebras of type F_4 and D_4 which do not contain a subalgebra of type G_2.

THEOREM 6. *Let k be a field of characteristic 0, $\mathfrak{L}$ an algebra of type Y_n and $\mathfrak{K}$ a subalgebra of $\mathfrak{L}$ of type X_l. Then $\mathfrak{L}$ is split if and only if $\mathfrak{K}$ is split for the following choices of (X_l, Y_n):*

(a) (G_2, D_{4I})
(b) (G_2, F_4)
(c) (D_{4I}, F_4)
(d) (D_{4I}, E_{6I})
(e) (F_4, E_{6I}) .

Proof. It is clear that a subalgebra of $\mathfrak{L}(\mathfrak{J})$, of type D_{4I} or F_4, is split if and only if $\mathfrak{L}(\mathfrak{J})$ is split. Thus (d) and (e) follow directly from Theorem $4I$(c) and Theorem $4I$(a) respectively, (c) from (d) and (e), (b) from (e) and Corollary 2 to Theorem 5 while (a) follows from (d) and Theorem 4II(a).

The restrictions imposed on the E_6 and D_4 forms in the above theorem are necessary (the Steinberg algebras of types D_{4II}, D_{4III}, D_{4VI}, and E_{6II} provide counterexamples).

THEOREM 7. *Let k, $\mathfrak{L}$ and $\mathfrak{K}$ be as in Theorem 6 and let α be an automorphism of $\mathfrak{K}$. Then α extends to an automorphism of $\mathfrak{L}$ for the following pairs (X_l, Y_n):*

(a) (G_2, D_4)
(b) (G_2, F_4)
(c) (D_4, E_{6I})
(d) (F_4, E_6) .

Proof. In (a) and (b) we may assume $\mathfrak{L} \subseteqq \mathfrak{D}(\mathfrak{J}) \subseteqq \mathfrak{L}(\mathfrak{J})$ for some reduced $\mathfrak{J}$ (Theorem 4II(a) and Lemma 1). A close examination of the proof of Theorem 5 shows that in this case α is the restriction of $X \rightarrow C^{-1}XC$ to $\mathfrak{K}$ where $C \in GL(\mathfrak{J})$ fixes $\mathfrak{J}_0(\mathfrak{K})$ pointwise (see (5)) hence $C \in \mathrm{Aut}_k \mathfrak{J}$. This establishes (b). Since $\mathfrak{J}_0(\mathfrak{L}) \subseteqq \mathfrak{J}_0(\mathfrak{K})$ (a) follows from [2], p. 258. (c) is obtained by a slight modification of the proof of Theorem 7 in [2]. In (d), the proof of Theorem 4I(a) shows that $\mathfrak{L} = \mathfrak{K} \oplus \sqrt{\lambda}\, R$ ($\lambda = 1$ for E_{6I}) where

$$\mathfrak{K} = \mathfrak{D}(\mathfrak{J}) \quad \text{and} \quad R = \{R_a \mid a \in \mathfrak{J},\ T(a) = 0\} .$$

If $\alpha \in \mathrm{Aut}\, \mathfrak{D}(\mathfrak{J})$ is realized as $D \rightarrow A^{-1}DA$, $A \in \mathrm{Aut}\, \mathfrak{J}$([21]) then $D + \sqrt{\lambda} R_a \rightarrow D^{\alpha} + \sqrt{\lambda} R_{aA}$ extends α.

One can sharpen the proofs of (a), (c), and (d) and show that isomorphism between subalgebras $\mathfrak{K}_i$ extends to an automorphism of $\mathfrak{L}$. Thus in these cases conjugacy sub-classes and isomorphism sub-classes are the same. In (b) this is not the case, as there is exactly one isomorphism class —determined by $\mathfrak{C}$ where $\mathfrak{L} = \mathfrak{D}(\mathfrak{h}(\mathfrak{C}_3, \gamma))$— but the conjugacy classes are represented by the isomorphism classes of annihilated subalgebras $\mathfrak{h}(k_3, \delta) \subseteqq \mathfrak{J}$. Thus there is a one-to-one correspondence between the conjugacy classes of G_2 subalgebras of $\mathfrak{D}(\mathfrak{J})$ and the equivalence classes of quadratic forms $\sum_{i=1}^{3} \gamma_i X_i^2$ such that $\mathfrak{J} \cong \mathfrak{h}(\mathfrak{C}_3, \gamma)$, $\gamma = \mathrm{diag}\,\{\gamma_1, \gamma_2, \gamma_3\}$.

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