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EXTREMELY AMENABLE ALGEBRAS

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Let S be a semigroup and $m(S)$ the space of bounded real functions on S. A subalgebra of $m(S)$ is extremely left amenable (ELA) if it is (sup) norm closed, left translation invariant, containing constants and has a multiplicative left invariant mean. S is ELA if $m(S)$ is ELA. In this paper, we give a method in constructing all ELA subalgebras of $m(S)$; it turns out that any such subalgebra of $m(S)$ is contained in an ELA subalgebra which is the uniform limit of certain classes of simple functions on S.

A subset $E \subseteqq S$ is left thick if for any finite subset $\sigma \subseteqq S$, there exists $s \in S$ such that $\{as; a \in \sigma\} \subseteqq E$. In §3, we strengthen a result of T. Mitchell and prove that a semigroup S is ELA if and only if for any subset $E \subseteqq S$, either E is left thick or $S - E$ is left thick. We also show how this result may be generalized to certain subalgebras of $m(S)$.

ELA semigroups and subalgebras have been considered by Mitchell in [9] and [10], and Granirer in [5], [6] and [7]. ELA semigroups S are shown to be characterized by the fixed point property on compact hausdorff spaces by Mitchell [9] and by the algebraic property: "for any a, b in S, there is a c in S such that $ac = bc = c$" by Granirer [5]. ELA subalgebras are characterized by Mitchell [10] by a fixed point property on compacta (under certain kinds of actions of S on a compact hausdorff space).

1. Some notations and preliminaries.

Let S be a semigroup. For each $a \in S, f \in m(S)$, denote by the sup norm of f, $||f|| = \sup_{s \in S}|f(s)|$ (and it is only this norm that will be used throughout this paper), ${}_af(s) = f(as)$ and $p_a(f) = f(a)$ for all $s \in S$. Then p_a is called the *point measure* on $m(S)$ at a and any element in $\mathrm{Co}\{p_a; a \in S\}$ is called a *finite mean* on $m(S)$ (where $\mathrm{Co}\,A$ denotes the convex hull of a subset A in a linear space).

If A is a norm closed left translation invariant subalgebra of $m(S)$ (i.e., ${}_af \in A$ whenever $f \in A$ and $a \in S$) containing 1, the constant one function on S, and $\varphi \in A^*$, then φ is a *mean* if $\varphi(f) \geqq 0$ for $f \geqq 0$, and $\varphi(1) = 1$; φ is *multiplicative* if $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in A$; φ is *left invariant* if $\varphi({}_sf) = \varphi(f)$ for all $s \in S$ and $f \in A$; and φ is a *point measure* [*finite mean*] on A if φ is the restriction of some point measure [finite mean] on $m(S)$ to A. It is well-known that the set of [point measure] finite mean on A is w^*-dense (i.e., $\sigma(A^*, A)$

-dense) in the set of [multiplicative] means on A. Furthermore, the set of multiplicative means on $m(S)$ is precisely βS, the Stone-Čech compactification of S ([3], p. 276).

A subalgebra of $m(S)$ is [*extremely*] *left amenable*, sometimes denoted by [ELA] LA, if it is norm closed, left translation invariant, containing constants and has a [multiplicative] left invariant mean (LIM). A semigroup S is [ELA] LA if $m(S)$ is [ELA] LA.

For any subset $E \subseteqq S$, $a \in S$, we shall denote by $\bar{E}$ = the closure of E in βS, $a^{-1}E = \{s \in S; as \in E\}$, $1_E \in m(S)$ such that $1_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{if } s \notin E \end{cases}$ and $\varphi(E) = \varphi(1_E)$ for any $\varphi \in m(S)^*$.

A subset $E \subseteqq S$ is *left thick* if for any finite subset $\sigma \subseteqq S$, there exists $s \in S$ such that $\{as; a \in \sigma\} \subseteqq E$, or equivalently, the family $\{s^{-1}E; s \in S\}$ has finite intersection property. Left thick subsets are first considered by Mitchell in [11]. Clearly, any left ideal of a semigroup is left thick. If S is left amenable, then every right ideal I is left thick, since if φ is a LIM on $m(S)$, then $\varphi(I) = 1$; consequently, the family $\{s^{-1}I; s \in S\}$ has finite intersection property.

2. The class of extremely amenable subalgebras. For any semigroup S, and $\mathscr{T}$ an algebra of subsets of S (i.e., a collection of subsets of S containing S and which are closed under complementation and finite union), we shall denote by

$$m(\mathscr{T}, S) = \text{norm closure of the linear span of the set } \{1_E; E \in \mathscr{T}\}\ .$$

Then $m(\mathscr{T}, S)$ is a norm closed subalgebra of $m(S)$ containing constants. Furthermore, if μ is a mean on $m(S)$, denote by

$$\mathscr{E}_\mu = \{E \subseteqq S; \mu(s^{-1}E) = \mu(E) = 1 \text{ for all } s \in S\}$$
$$\mathscr{T}_\mu = \text{algebra generated by } \mathscr{E}_\mu\ .$$

REMARK 1. For any semigroup S:

(a) $\mathscr{E}_\mu$ is nonempty for all mean μ on $m(S)$ since $S \in \mathscr{E}_\mu$.

(b) If μ is a multiplicative LIM on $m(S)$, then $m(\mathscr{T}_\mu, S) = m(S)$.

(c) If S has f.i.p.r.i. (finite intersection for right ideals), $\mu \in \bigcap\{\overline{sS}; s \in S\}$ then $aS \in \mathscr{E}_\mu$ for all $a \in S$. In particular, all right ideals of S are left thick. To see this we only have to observe that for each $a, t \in S$, $t^{-1}(aS) \supseteqq bS$ where b is chosen such that $tb \in aS$. Conversely, if all right ideals of a semigroup S are left thick, then S has f.i.p.r.i. since for any $a, b \in S$, there exists $c \in S$ such that $bc \in aS$.

(d) If S generates a group G, S has f.i.p.r.i. and $\mu \in \bigcap\{\overline{sS}; s \in S\}$, where the closure is taken in βG, then $gS \in \mathscr{E}_\mu$ for all $g \in G$. In fact, for any $g \in G$,, gS contains a right ideal of S ([12], Lemma 5.1) and hence $\mu(g_1^{-1}(g_2S)) = 1$ for all $g_1, g_2 \in G$. In particular, each gS (and therefore S) is a left thick subset of G, $g \in G$.

Our first main result is to show that for any semigroup S, every ELA subalgebra of $m(S)$ is contained in an ELA subalgebra $m(\mathscr{T}_\mu, S)$ for some mean μ on $m(S)$. We shall prove this result in a series of lemmas.

LEMMA 1. *Let S be a semigroup, $F \subseteqq m(S)$ such that $_sf \in F$ for all $f \in F$ and $s \in S$. If A is the smallest norm closed subalgebra containing F and the constant functions, then A is left translation invariant. If $\varphi(_sf) = \varphi(f)$, $\varphi \in \beta S$, for all $s \in S$ and $f \in F$, then φ is a multiplicative* LIM *on A.*

Proof. A is the norm closure of H, where H consists of all functions of the form $a_0 1 + a_1 g_1 + \cdots + a_n g_n$ and for each $i = 1, \cdots, n$, g_i is a finite product of functions in F. Then as readily checked, $_sh \in H$ for all $s \in S$ and $h \in H$. If $f \in A$, and $h_\alpha \in H$ such that $\lim_\alpha \| h_\alpha - f \| = 0$, then $\lim_\alpha \| {_sh_\alpha} - {_sf} \| \leqq \lim_\alpha \| h_\alpha - f \| = 0$, and hence $_sf \in A$ for all $s \in S$. The last assertion can be proved similarly.

LEMMA 2. *Let S be a semigroup, $f \in m(S)$ and $\varphi \in \beta S$ be such that $\varphi(_sf) = \varphi(f)$ for all $s \in S$;*

(a) *if $\varphi(f) \neq 0$, then $\{s \in S; f(s) \neq 0\} \in \mathscr{E}_\varphi$*

(b) *if $\varphi(f) = 0$, then $\{s \in S; f(s) < c\} \in \mathscr{E}_\varphi$ for all $c > 0$.*

Proof. (a) If $N = \{s \in S; f(s) \neq 0\}$, then $\varphi(f) = \varphi(1_N f) = \varphi(1_N)\varphi(f)$ and $\varphi(_sf) = \varphi(_s(1_N f)) = \varphi(1_{s^{-1}N})\varphi(_sf)$. Hence $\varphi(N) = \varphi(s^{-1}N) = 1$ for all $s \in S$.

(b) Let A be the smallest norm closed subalgebra containing f and all its left translates and constants. Then as well-known, A is a lattice [2]. Define $h(s) = \max\{c - f(s), 0\}$, then $h \in A$, and $\varphi(_sh) = \varphi(h)$ for all $s \in S$ (Lemma 1). Since $\varphi(h) \geqq c > 0$, it follows from (a) that $\{s \in S; h(s) \neq 0\} = \{s \in S; f(s) < c\} \in \mathscr{E}_\varphi$.

LEMMA 3. *For any semigroup S and $E \subseteqq S$, if E is left thick, then there exists $\varphi \in \beta S$ such that $\varphi(s^{-1}E) = \varphi(E) = 1$ for all $s \in S$.*

Proof. Let $\psi \in \bigcap_{s \in S} \overline{s^{-1}E}$ and define $\varphi \in \beta S$ by $\varphi(f) = \psi(h)$ where $h(s) = \psi(_sf)$ for all $s \in S$.

THEOREM 1. *Let S be a semigroup and A be a norm closed left translation invariant subalgebra of $m(S)$ containing constants, then*

A is ELA *if and only if* $A \subseteqq m(\mathscr{T}_\mu, S)$ *for some [multiplicative] mean* μ *on* $m(S)$.

Proof. For any mean μ on $m(S)$, $m(\mathscr{T}_\mu, S)$ is the smallest norm closed subalgebra containing $F = \{1_E; E \in \mathscr{E}_\mu\}$ and constants. It follows from Lemma 1 that $m(\mathscr{E}_\mu, S)$ is necessarily left translation invariant. Furthermore, any $\varphi \in \bigcap_{E \in \mathscr{E}_\mu} \bar{E}$ (which is nonempty by compactness of βS) is a multiplicative LIM on $m(\mathscr{T}_\mu, S)$ since $\varphi(E) = \varphi(s^{-1}E) = 1$ for all $s \in S$ and $E \in \mathscr{E}_\mu$ (Lemma 1). Consequently, the restriction of φ to A is a multiplicative LIM.

Conversely, if A has a multiplicative LIM ψ, and $\{p_{a_\alpha}\}$, $a_\alpha \in S$, is a net of point measure on $m(S)$ such that $\lim_\alpha p_{a_\alpha}(f) = \psi(f)$ for all $f \in A$, then any cluster point μ of the net $\{p_{a_\alpha}\}$ in βS is a multiplicative extension of ψ to $m(S)$. Let $I = \{f \in A; \mu(f) = 0\}, f \in I$ be arbitrary and $\lambda > 0$. For each $n \in Z$, the integers, define

$$K(n, \lambda) = \{s \in S; \lambda n \leqq f(s) < \lambda(n+1)\} .$$

Then $S - K(n, \lambda) \in \mathscr{E}_\mu$ for all $n \in Z - \{-1, 0\}$ (by Lemma 2b) and $\|f - \sum(\lambda n)1_{K(n,\lambda)}\| \leqq \lambda$, where the sum is taken over all $n \in Z - \{-1, 0\}$. Thus $A = I \oplus C \subset m(\mathscr{T}_\mu, S)$, where C is the algebra of constant functions, since $m(\mathscr{T}_\mu, S)$ is closed in $m(S)$.

REMARK 2. If S is endowed with a noncompact hausdorff topology such that for each compact subset $\sigma \subseteqq S$, $s^{-1}\sigma$ is compact for all $s \in S$; order $E = \{\sigma; \sigma$ compact subset of $S\}$ by upward inclusion. For $\sigma \in E$, let $a_\sigma \in S - \sigma$. If μ is a cluster point of the net of point measures $\{p_{a_\sigma}; \sigma \in E\}$, then for any $\sigma \in E$, $\mu(s^{-1}(S - \sigma)) = \mu(S - \sigma) = 1$ for all $s \in S$. Hence, $S - \sigma$ is left thick for all compact subsets $\sigma \subseteqq S$ and the ELA subalgebra $m(\mathscr{T}_\mu, S)$ includes all functions $f \in m(S)$ which vanish at infinity. In fact for any such f (fixed but arbitrary), let $\lambda > 0$. For each $n \in Z$, the integers, define

$$K(n, \lambda) = \{s \in S; \lambda n \leqq f(s) < \lambda(n+1)\} .$$

Since each $S - K(n, \lambda)$ is included in a compact subset of S, $K(n, \lambda) \in \mathscr{T}_\mu$ for all $n \in Z$ and

$$\|f - \sum_{n \in N}(\lambda n)1_{K(n,\lambda)}\| < \lambda .$$

Theorem 1 yields the following consequence:

COROLLARY. *For any semigroup* S, $m(S)$ *has a nontrivial* ELA *subalgebra (i.e., other than the algebra of constant functions) if and only if* S *has a proper left thick subset.*

Proof. If S has a proper left thick subset E, let μ be a multiplicative mean on $m(S)$ such that $\mu(E) = \mu(s^{-1}E) = 1$ for all $s \in S$ (Lemma 3), then $E \in \mathscr{E}_\mu$, and $m(\mathscr{T}_\mu, S)$ is a nontrivial ELA subalgebra of $m(S)$ (Theorem 1). Conversely, if A is a nontrivial ELA subalgebra of $m(S)$, then $A \subseteqq m(\mathscr{T}_\mu, S)$ for some mean μ on $m(S)$. Consequently, $m(\mathscr{T}_\mu, S)$ is nontrivial and hence $\mathscr{E}_\mu$ contains a proper subset of S, which is necessary left thick.

REMARK 3. The class of semigroups S for which $m(S)$ has a nontrivial ELA subalgebra is extremely big and they include semigroups S which satisfy any one of the following conditions:

(a) S is finite and not right cancellative.

(b) S is infinite and left cancellative.

(c) S is infinite and has finite intersection property for right ideals (note that any left amenable semigroup has the latter property).

(d) S has finite intersection property for left ideals and the factor semigroup $S|(\swarrow)$ is infinite, where $(\swarrow)$ is the two-sided stable equivalence relation defined by $a(\swarrow)b$ if and only if $ca = cb$ for some $c \in S$ (an equivalence relation E on S is two-sided stable if aEb implies $acEbc$ and $caEcb$ for all $c \in S$).

In fact, we only need to show that the semigroups listed in (a), (b), (c) and (d) have proper left thick subsets. (a) If $a, b, c \in S$ are such that $a \neq b$ and $ac = bc$, then Sc is a proper left thick subset in S. (b) It follows from Remark 2 (with the discrete topology) that for any finite subset $\sigma \subseteqq S$, $S - \sigma$ is left thick. (c) We may assume that S is not cancellative (for otherwise (b) shows that S has a proper left thick subset); then S has either a proper left ideal or a proper right ideal, which must be left thick (Remark 1(c)). (d) The factor semigroup $S/(\swarrow)$ if left cancellative ([4], p. 372). It follows from (b) that $S|(\swarrow)$ has a proper left thick subset $\tilde{A}$. If $A = \{s; \bar{s} \in \tilde{A}\}$, where $\bar{s}$ denotes the homomorphic image of s in $S/(\swarrow)$, then A is a proper left thick subset in S.

Examples of semigroups S for which the only ELA subalgebra of $m(S)$ is the algebra of constant functions include all semigroups of the form $E' \times G'$ where E' is a left zero semigroup (i.e., $a \cdot b = a$ for all $a, b \in E'$) and G' is a finite right cancellative semigroup as the following proposition shows:

PROPOSITION 1. *The following conditions concerning a semigroup S are equivalent*:

(a) *S is right cancellative and has no proper left thick subset.*

(b) *S has an idempotent and has no proper left thick subset.*

(c) *S is the direct product $E \times G$ of a finite group G and a left zero semigroup E.*

(d) *S is the direct product $E' \times G'$ of the finite right cancellative semigroup G' and a left zero semigroup E'.*

Proof. (a) implies (b) follows from theorem 1.2.7 in [13] (p. 38). If (b) holds, the same theorem in [13] shows that S is the direct product $E \times G$ of a group G and a left zero semigroup E. G is finite, for otherwise G has a proper left thick subset T (Remark 3(b)) which implies that S has a proper left thick subset $E \times T$. (c) implies (d) is clear. Finally if (d) holds, then as readily checked, S is right cancellative. Finally if K is a left thick subset in S, $t \in E'$ is arbitrary, there exists $(t_0, g_0) \in E' \times G'$ such that $\{(tt_0, gg_0); g \in G\} = \{(t, g); g \in G\} \subseteqq K$. Consequently, $K = S$.

3. A characterization theorem. Mitchell ([9], Th. 1) shows that a semigroup S is ELA if and only if for each finite collection of subsets $E_i \subseteqq S$, $i = 1, \cdots, n$ such that $S = \bigcup_{i=1}^n E_i$, it follows that at least one of the subsets E_i is left thick in S. We show in this section that Mitchell's result can be sharpened and generalized to certain subalgebras of $m(S)$. Our proof is completely different from that of Mitchell [9].

THEOREM 2. *For any semigroup S, and $\mathscr{T}$ an algebra of subsets of S such that $s^{-1}E \in \mathscr{T}$ for all $s \in S$ and $E \in \mathscr{T}$, the following conditions are equivalent*:

(a) $m(\mathscr{T}, S)$ is ELA.

(b) *For each finite collection $\{E_1, \cdots, E_n\}$ of disjoint sets from $\mathscr{T}$ with union S, at least one of E_i is left thick.*

Proof. (a) $\Rightarrow$ (b) Let φ be a multiplicative LIM on $m(\mathscr{T}, S)$, then $1 = \varphi(S) = \sum_{i=1}^n \varphi(E_i)$. Hence $\varphi(E_i) > 0$ for some i, which implies $\varphi(s^{-1}E_i) = \varphi(E_i) = 1$ for all $s \in S$, since φ is multiplicative. Consequently, the family $\{s^{-1}E_i; s \in S\}$ has finite intersection property, and hence E_i is left thick.

(b) $\Rightarrow$ (a) Let $\mathscr{P}$ be the set each of whose elements is a finite collection $\{E_1, \cdots, E_n\}$ of disjoint sets in $\mathscr{T}$ with union S. Let $\mathscr{P}$ be ordered by defining $P_1 \leqq P_2$ to mean that each set in P_1 is the union of sets in P_2, $P_1, P_2 \in \mathscr{P}$. It is easy to see that $\leqq$ renders $\mathscr{P}$ into a directed set. For each $E \in \mathscr{T}$, let $K_E = \{\varphi \in \beta S; \varphi(s^{-1}E) = \varphi(E)$ for all $s \in S\}$. K_E is a nonempty and closed subset of βS, and the family $\{K_E; E \in \mathscr{T}\}$ has the finite intersection property. In fact, if $E_1, \cdots, E_n \in \mathscr{T}$, let $P_i = \{E_i, S - E_i\} \in \mathscr{P}$, and choose $P_0 \in \mathscr{P}$ such that $P_0 \geqq P_i$ for each $1 \leqq i \leqq n$. By assumption, there exists F in P_0 such that F is left thick. Let $\varphi_0 \in \beta S$ such that $\varphi_0(s^{-1}F) = \varphi_0(F) = 1$

for all $s \in S$ (Lemma 3). If $F \subseteqq E_i$, then $s^{-1}F \subseteqq s^{-1}E_i$ for all $s \in S$. Hence $\varphi_0(s^{-1}E_i) = \varphi_0(E_i) = 1$ for all $s \in S$. If $F_0 \subseteqq S - E_i$, then $\varphi_0(s^{-1}(S - E_i)) = \varphi_0(S - E_i) = 1$ for all $s \in S$. Consequently, $\varphi_0(s^{-1}E_i) = \varphi_0(E_i) = 0$ for all $s \in S$. Hence $\varphi_0 \in \bigcap_{i=1}^{n} K_{E_i}$. If $\varphi \in \bigcap_{E \in \mathscr{T}} K_E$ (which is nonempty by compactness of βS), then $\varphi(s^{-1}E) = \varphi(E)$ for all $s \in S$ and $E \in \mathscr{T}$. Consequently, φ is a multiplicative LIM on $m(\mathscr{T}, S)$ (Lemma 1).

LEMMA 4. *A semigroup S is* ELA *if and only if for each subset $E \subseteqq S$, there exists a mean μ_E on $m(S)$ such that $\mu_E(s^{-1}E) = \mu_E(E) \in \{0, 1\}$ for all $s \in S$.*

Proof. If φ is a multiplicative LIM on $m(S)$, then for any subset $E \subseteqq S$, $\varphi(E)$ is either 0 or 1. To see the converse, for each $E \subseteqq S$, let $K_E = \{\varphi \in \beta S; \varphi(s^{-1}E) = \varphi(E)$ for all $s \in S\}$. Then K_E is nonempty since if $\mu_E(s^{-1}E) = \mu_E(E) = 1$ for all $s \in S$, then $\mu_E(s^{-1}E \cap E) = 1$ for all $s \in S$ and hence the family $\{\overline{s^{-1}E \cap E}: s \in S\}$ has finite intersection property. Let $\varphi \in \bigcap_{s \in S} \overline{s^{-1}E \cap E}$, then $\varphi(s^{-1}E) = \varphi(E) = 1$ for all $s \in S$. If $\mu_E(s^{-1}E) = \mu_E(E) = 0$ for all $s \in S$, then $\mu_E(s^{-1}(S - E)) = \mu_E(S - E) = 1$ for all $s \in S$. Hence as above, there exists $\varphi \in \beta S$ such that $\varphi(s^{-1}(S - E)) = \varphi(S - E) = 1$ for all $s \in S$, or $\varphi(s^{-1}E) = \varphi(E) = 0$ for all $s \in S$. In both cases, $K_E \neq \varnothing$. Furthermore, $\mathscr{K} = \{K_E; E \subseteqq S\}$ is a family of nonempty w^*-compact subset of βS. If we can show that $\mathscr{K}$ has the finite intersection property, then any $\varphi \in \bigcap_{E \subseteqq S} K_E$ satisfies $\varphi(s^{-1}E) = \varphi(E)$ for all $s \in S$ and $E \subseteqq S$. By Lemma 1, φ is even a LIM on $m(S)$. To this end, let $\mathscr{E}$ be a family of subsets of S such that $\bigcap_{E \in \mathscr{E}} K_E \neq \varnothing$, and let $E_0 \subseteqq S$. Pick $\varphi \in \bigcap_{E \in \mathscr{E}} K_E$ and $\mu \in K_F$ where $F = \{s \in S; \varphi(s^{-1}E_0) = 1\}$. Define $\psi \in \beta S$ by $\psi(f) = \mu(h)$, where $h(s) = \varphi({}_sf)$ for all $s \in S$. Then $\psi \in (\bigcap_{E \in \mathscr{E}} K_E) \cap K_{E_0}$ since $\psi(E) = \mu(h) = \mu({}_ah) = \psi(a^{-1}E)$ for all $a \in S$, where $h(s) = \varphi(s^{-1}E) = \varphi(s^{-1}(a^{-1}E)) = h(as)$ for all $a, s \in S$, and

$$\psi(E_0) = \mu(F) = \mu(a^{-1}F) = \psi(a^{-1}E_0) \text{ for all } a \in S .$$

This finishes the proof.

Lemma 4 yields the following new characterization theorem for the class of ELA semigroups:

THEOREM 3. *A semigroup S is* ELA *if and only if* (*) *for each subset $E \subseteqq S$, either E is left thick, or $S - E$ is left thick.*

Proof. Necessity follows from Theorem 2 (a) $\Rightarrow$ (b). Conversely if (*) holds, it follows from Lemma 3 that for each $E \subseteqq S$, there exists a mean μ on $m(S)$ such that $\mu(s^{-1}E) = \mu(E) = 1$ for all $s \in S$

if E is left thick, or $\mu(s^{-1}E) = \mu(E) = 0$ if $S - E$ is left thick. Consequently, S is ELA by Lemma 4.

REMARK. Note that condition (*) in Theorem 3 is formally weaker than condition (b) and (c) in [9], Theorem 1.

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