PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 3, 1970

WILD POINTS OF CELLULAR ARCS IN 2-COMPLEXES IN E^3 AND CELLULAR HULLS

GAIL H. ATNEOSEN

Loveland has established that if W is the set of wild points of a cellular arc that lies on a 2-sphere in E^3, then either W is empty, W is degenerate, or W contains an arc. This note considers 2-complexes rather than 2-spheres. Making strong use of Loveland's results and others, it is proved that a cellular arc in a 2-complex in E^3 either contains an arc of wild points or has at most one wild point that has a neighborhood in the 2-complex homeomorphic to an open 2-cell. In the case of noncellular arcs in E^3, one can investigate "minimal cellular sets" containing the arc. A *cellular hull* of a subset A of E^3 is a cellular set containing A such that no proper cellular set also contains A. A characterization is given of those arcs in E^3 that have cellular hulls that lie in tame 2-complexes in E^3.

A *2-complex* in E^3 is the homeomorphic image of a 2-dimensional finite Euclidean polyhedron. A subset X of E^3 is said to be *locally tame* at a point p of X if there is a neighborhood N of p in E^3 and a homeomorphism h of $\mathrm{Cl}(N)$ (Cl = closure) onto a polyhedron in E^3 such that $h(\mathrm{Cl}(N \cap X))$ is a finite Euclidean polyhedron. A point p of a subset X of E^3 is said to be a *wild point* of X if X is not locally tame at p. A subset G of E^3 is said to be *cellular* (in E^3) if there exists a sequence $Q_1, Q_2, \cdots$ of 3-cells in E^3 such that for each positive integer i, $Q_{i+1} \subset \mathrm{Interior}\ Q_i$ and $G = \bigcap_{i=1}^{\infty} Q_i$. If A and B are two arcs in E^3, then A is said to be *equivalent* to B if there is a homeomorphism h mapping E^3 onto E^3 such that $h(A) = B$.

THEOREM 1. *Let A be a cellular arc in a 2-complex in E^3. If the set of wild points of A does not contain an arc, then A has at most one wild point that has a neighborhood in the 2-complex homeomorphic to an open 2-cell.*

Proof. Assume that A has two wild points p and q that have neighborhoods in the 2-complex homeomorphic to an open 2-cell and contradict the hypothesis that A is cellular. Then p lies on a subarc of A that is contained in the interior of a closed 2-cell. The argument of Theorem 5 of [3] then establishes that p lies on a subarc C of A that is contained in a 2-sphere in E^3. Since C is a cellular arc by [6], it follows from [5] that p is the only wild point of C. Thus p and q are isolated wild points of A.

If p and q are the endpoints of A, it follows from Theorem 10 of [8] that A is not cellular, so this case cannot occur.

Next consider the case when p is an interior point of A and q is an endpoint of A. As above, we obtain that p lies interior to a subarc C of A whose only wild point is p and that C is contained in a 2-sphere S. By [4] and [2] we may assume that S is locally polyhedral except at p. If C_1 and C_2 are subarcs of C such that $C_1 \cup C_2 = C$ and $C_1 \cap C_2 = p$, then Theorem 5 of [4] implies that C_1 and C_2 are equivalent. An application of Theorem 1 of [4] yields that if C_1 and C_2 are both locally tame at p then C is locally tame at p. Hence p is a wild point of both C_1 and C_2. Let B be a subarc of A with endpoints p and q. Then B is a cellular arc whose endpoints are isolated wild points, by [8] this case cannot occur.

By arguments as in the above two cases, it follows that the last case, in which both p and q are interior points of A, can also not occur.

For the following theorem we need to define a particular 2-complex called a 3-book. A 3-*book* is defined to be a subset of E^3 which is the union of three closed 2-cells which meet precisely on a single arc on the boundary of each.

THEOREM 2. *An arc A in E^3 has a cellular hull that lies in a tame 2-complex in E^3 if and only if A is equivalent to an arc in a tame 3-book.*

Proof. If A has a cellular hull that lies in a tame 2-complex, then the set of wild points of A is a closed totally disconnected set. It follows easily from [7] that such an arc is equivalent to an arc in a tame 3-book.

Conversely, suppose that A lies in a tame 3-book B. Consider a maximal chain (ordered by inclusion) that has B as a member and also has the property that each member of the chain is a cellular set that contains A. The intersection of the members of this maximal chain then yields a cellular hull of A that lies in the tame 2-complex B.

The arc in [1] is an example of an arc that does not have a cellular hull that lies in any tame 2-complex.

REFERENCES

1. W. R. Alford, *Some "nice" wild 2-spheres in E^3*, Topology of 3-manifolds and related topics, (Proc. The Univ. of Georgia Institute, 1961), 29-33. Prentice-Hall,

Englewood Cliffs, N. J., 1962.
2. R. H. Bing, *Locally tame sets are tame,* Ann. of Math. **59** (1954), 145-158.
3. ———, *A surface is tame if its complement is* 1-*ULC*, Trans. Amer. Math. Soc. **101** (1961), 294-305.
4. P. H. Doyle and J. G. Hocking, *Some results on tame disks and spheres in* E^3, Proc. Amer. Math. Soc. **11** (1960), 832-836.
5. L. D. Loveland, *Wild points of cellular subsets of* 2-*spheres in* S^3, Michigan Math. J. **14** (1967), 427-431.
6. D. R. McMillan, *A criterion for cellularity in a manifold,* Ann. of Math. **79** (1964), 327-337.
7. E. E. Posey, *Proteus forms of wild and tame arcs,* Duke Math. J. **31** (1964), 63-72.
8. C. D. Sikkema, *A duality between certain spheres and arcs in* S^3, Trans. Amer. Math. Soc. **22** (1966), 339-415.

Received October 22, 1969.

WESTERN WASHINGTON STATE COLLEGE