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A NOTE ON THE CHARACTERIZATION OF CONDITIONAL EXPECTATION OPERATORS

DANIEL E. WULBERT

Let $(X, \sum, \mu)$ be an arbitrary measure space. A complete characterization is presented for the norm one positive projections P of $L^1(X, \sum, \mu)$ into itself such that $||f||_\infty \geqq ||Pf||_\infty$ for each essentially bounded, summable function f.

If $(X, \sum, \mu)$ is a probability measure space it is known [1], [2], and [4] that such operators coincide precisely with the conditional expectation operators defined on L^1 (see definitions below). In this note we show that this characterization extends to an arbitrary measure space. The proof presented here is a direct, constructive proof requiring only basic measure theory. Although this extension is not unexpected, it does not seem to be a consequence of the methods used in the case for the finite measure spaces. An independent proof for this result, using the ergodic theory of Markov processes, was found by S. Foguel. Also extensions to arbitrary measure spaces of related theorems in [1], have recently been done by L. Tzafriri [5].

DEFINITION. Let $(X, \sum, \mu)$ be a measure space. Let $\sum_0$ be a σ-subring of $\sum$. We call a projection P on $L^1(X, \sum, \mu)$ a *conditional expectation operator* with respect to $\sum_0$ if Pf is $\sum_0$ measurable for all f, and if for each U in $\sum_0$ we have

$$\int_U Pfd\mu = \int_U fd\mu .$$

Clearly a conditional expectation operator is positive projection of norm one.

Notation. Let P be a norm one positive projection from $L^1(X, \sum, \mu)$ onto E. Let

$$\textstyle\sum_0 = \{K \subseteqq X : K = \operatorname{supp} f, f \geqq 0, f \text{ in } E\} .$$

We use $\operatorname{supp} f$ to denote the support of a function f. The characteristic function of a set A is written 1_A.

LEMMA 1. *E is a lattice.*

Proof. Let f and g be in E. Since $f \vee g$ dominates f and g, $P(f \vee g)$ dominates f, g and hence $f \vee g$. Since $|| f \vee g || \geqq || P(f \vee g)||$, we have that $f \vee g = P(f \vee g)$.

LEMMA 2. *$\sum_0$ is a σ-ring.*

Proof. We need to show that countable unions of members of $\sum_0$ are in $\sum_0$, and differences of members in $\sum_0$ are in $\sum_0$. Suppose U_i is in $\sum_0$ for $i = 1, 2, \cdots$. Let f_i be norm one positive functions in E whose support in U_i. Then $f = \sum_{i=1}^{\infty} (1/2)^i f_i$ is a positive function in E whose support is $\bigcup_{i=1}^{\infty} U_i$.

Suppose U and V are in $\sum_0$, and are the supports of positive fuction f and g in E.
Let

$$f_n = (f - ng) \vee 0 .$$

Let

$$f'(x) = \begin{cases} f(x) & \text{for } x \text{ in } U - V \\ 0 & \text{otherwise} . \end{cases}$$

From the dominated convergence theorem f_n converges to f'. Thus $U - V$ is in $\sum_0$

LEMMA 3. *Suppose f vanishes off some member of $\sum_0$. The following is true.*

(i) *$P(|f|) = 0$ implies $f = 0$.*

(ii) *if K is in $\sum_0$ then $P(1_K f) = 1_K P(f)$.*

(iii) *$\int_K Pfd\mu = \int_K fd\mu$ for all K in $\sum_0$.*

Proof. We will assume that f is nonnegative. Suppose g is a nonnegative number of E such that f vanishes off $\operatorname{supp}(g)$. Since $f \wedge ng$ increases monotonically to f, it suffices to assume that f is bounded by a member of E. We will assume therefore that $0 \leqq f \leqq g$.

(i) $P(g - f) = g$ so $\| g \| \leqq \| g - f \|$, but $0 \leqq g - f \leqq g$. Therefore $f = 0$.

(ii) Suppose that h is a nonnegative member of E such that $\operatorname{supp} h = K$. Since $g \wedge nh$ converges monotonically to $1_K g$, it follows that $1_K g$ is in K.

Now $0 \leqq P(1_K f) \leqq P(1_K g) = 1_K g$. Hence $P(1_K f)$ vanishes off K, and $P(1_K f) \leqq 1_K P(f)$. We also have $1_K g - P(1_K f) = P(1_K(\dot{g} - f)) \leqq 1_K P(g - f) = 1_K g - 1_K P(f)$. Thus $1_K Pf = P(1_K f)$.

(iii) $\int_K fd\mu = \| 1_K f \| \geqq \| P(1_K f) \| = \int_K Pfd\mu$. Similarly

$$\int g d\mu - \int_K f d\mu = \int_K (g - f) d\mu = \| 1_K (g - f) \|$$
$$\geqq \| P(1_K(g - f)) \| = \int_K (Pg - Pf) d\mu .$$

Hence $\int_K Pf d\mu = \int_K f d\mu$.

For the remainder of the paper we will also assume that for each essentially bounded f in L^1, $\| f \|_\infty \geqq \| Pf \|_\infty$.

LEMMA 4. *Each member of E is $\sum_0$-measurable.*

Proof. We first show that $g \wedge c$ is in E for each constant function c and each g in E. It suffices to prove this assertion for positive functions g and for $c > 0$. However this is almost obvious for since P is positive $g \geqq P(g \wedge c) \geqq 0$, and from the hypothesis $c \geqq P(g \wedge c)$. Thus $g \wedge c \geqq P(g \wedge c)$. Now with $K = \operatorname{supp} g$ Lemma 3 (iii) implies that $g \wedge c = P(g \wedge c)$.

It follows that for any c, $g - g \wedge c$ is in E. Hence if g is a positive function in E, the set $\{x \text{ in } X : g(x) > c\}$ is also the support of $g - g \wedge c$, and thus is in $\sum_0$. Hence Pf is $\sum_0$ measurable for each f in L^1.

PROPOSITION. *P is the conditional expectation operator with respect to $\sum_0$.*

Proof. Let f be in L^1. Let $r = \sup\left\{\int_K | f | d\mu : K \text{ in } \sum_0\right\}$. Let K be a member of $\sum_0$ such that $\int_K | f | d\mu = r$. Writing $f = 1_K f + (f - 1_K f)$ we see that f is the sum of a function which vanishes off a member of $\sum_0$ and a function which vanishes on each member of $\sum_0$. Thus in view of all the previous lemmas it remains only to show that $Pf = 0$ if f vanishes on each member of $\sum_0$. We may assume that f is bounded and nonnegative. Since the support of nPf is in $\sum_0$, and since f vanishes on all members of $\sum_0$, we have

$$\| f + nPf \|_\infty = \max (\| f \|_\infty, n \| Pf \|_\infty) ,$$

but

$$\| f + nPf \|_\infty \geqq \|P(f + nPf) \|_\infty = (n + 1) \| Pf \|_\infty .$$

This implies that $\| Pf \|_\infty = 0$.

REMARKS. The hypothesis that $\| f \|_\infty \geqq \| Pf \|_\infty$ is equivalent to the assumption that $P(1_A) \leqq 1$ for all sets A of finite measure.

The referee has pointed out that the main result in this note is valied for norm one projections defined on L_p spaces. Besides the proofs presented here, one would also use the fact that there do not exist two distinct norm one projections of a smooth space onto a subspace. (For L_p spaces this result is in [1]. For smooth spaces a proof is in [6, Lemma 1]). The organization of this note was also suggested by the referee, and adapts to L_p operators more readily than the original.

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University of Washington
Seattle, Washington