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ON CERTAIN TOEPLITZ OPERATORS IN TWO VARIABLES

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The problem of inverting and/or factoring Weiner-Hopf operators in two variables is one of the basic unsolved problems in classical analysis. In this paper we shall consider operators which are a perturbation of a product of operators in one variable, the perturbation differing from such simple operators by an operator in one variable. The principal tools used are the spectral mapping theorem combined with the known results on operators in one variable.

2. **Preliminaries.** Consider the space l_2 of sequences of complex numbers

$$\xi = \{\xi_{ij}\}_{i,j}^{\infty} c_{ij} = 0$$

with

$$\|\xi\| = \left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\xi_{kj}|^2\right)^{1/2} < \infty \ . \tag{2.1}$$

Let

$$a = \{a_j\}_{j=-\infty}^{\infty} \qquad b = \{b_j\}_{j=-\infty}^{\infty} \qquad c = \{c_j\}_{j=-\infty}^{\infty} \tag{2.2}$$

be absolutely convergent sequences of complex numbers. Define

$$\begin{aligned} g &= \{g_{ij}\}_{i,j=-\infty}^{\infty} \\ g_{ij} &= a_i b_j + c_j \delta(j) \end{aligned} \tag{2.3}$$

for

$$\delta(j) = 0 \text{ if } j \neq 0 \ , \quad \delta(0) = 1 \ .$$

It is clear that

$$\sum_{j=-\infty}^{\infty}\ \sum_{i=-\infty}^{\infty} |g_{ij}| < \infty$$

We are concerned with the operation

$$T_g: l_2 \rightarrow l_2 \ ,$$

$$\begin{aligned} (T_g\xi)_{ij} &= \sum_{L=0}^{\infty}\ \sum_{k=0}^{\infty} g_{j-k,j-L}\xi_{kL} \\ [g_{ij} &= g_{i,j}] \ . \end{aligned} \tag{2.4}$$

Our techniques and results are exactly the same in the two variable Weiner-Hopf integral analogue.

Define

$$G(e^{i\theta}, e^{i\varphi}) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g_{kj} e^{-ij\varphi - ik\theta}$$

(2.5)
$$A(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\theta}$$
$$B(e^{i\varphi}) = \sum_{k=-\infty}^{\infty} b_k e^{-ik\varphi}$$
$$C(e^{i\varphi}) = \sum_{j=-\infty}^{\infty} c_j e^{-ij\varphi} .$$

So

(2.6) $$G(e^{i\theta}, e^{i\varphi}) = A(e^{i\theta})B(e^{i\varphi}) + C(e^{i\varphi}) .$$

Let $\hat{l}_2$ denote the space of doubly infinite sequences in the second subscript, singly infinite in the first.

$$\xi \in \tilde{l}_2 , \qquad \xi = \{\xi_{ij}\}_{j=-\infty, i=0}^{\infty}$$

(2.7) $$\| \xi \|_{\sim} = \Big(\sum_{j=-\infty}^{\infty} \sum_{i=0}^{\infty} | \xi_{ij} |^2 \Big)^{1/2} .$$

Let us Fourier transform l_2 and $\tilde{l}_2$ with respect to the second subscript, i.e.,

(2.8) $$\hat{\xi} = \{\xi_k(e^{i\varphi})\}_{k=0}^{\infty} \in E_2$$

where

$$\xi_k(e^{i\varphi}) = \sum_{j=0}^{\infty} \xi_{kj} e^{ij\varphi} .$$

We shall obtain on E_2 a transformed Toeplitz operator whose elements are themselves Toeplitz operators

Define

(2.9) $$P_+^{(\varphi)} \sum_{j=-\infty}^{\infty} \xi_{kj} e^{ij\varphi} = \sum_{j=0}^{\infty} \xi_{kj} e^{ij\varphi} ,$$

and on the space of singly semi-infinite sequences, define the Toeplitz map

(2.10) $$(A\eta)_i = \sum_{j=0}^{\infty} a_{i-j} \eta_j .$$

Thus, the total operator transforms to the compound operator

(2.11) $$P_+^{(\varphi)} B(e^{i\varphi}) A \hat{\xi} + P_+^{(\varphi)} C(e^{i\varphi}) \hat{\xi} = P_+^{(\varphi)} L(e^{i\varphi}) \hat{\xi} .$$

3. Main results.

MAIN THEOREM. (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C)

(A) (1) $G(e^{i\theta}, e^{i\varphi}) \neq 0$ real θ, φ.

(2) The change in argument of $G(e^{i\theta}, e^{i\varphi})$ as θ goes from 0 to 2π is 0 for any real φ.

(3) The change in argument of $G(e^{i\theta}, e^{i\varphi})$ as φ goes from 0 to 2π is 0 for any real θ.

(B) $L(e^{i\varphi})$ can be factored

$L(e^{i\varphi}) = L_-(e^{i\varphi})L_+(e^{i\varphi})$ for $0 \leqq \varphi \leqq 2\pi$ where

$L_-(e^{i\varphi})$, $L_+(e^{i\varphi})$ commute and are continuous in φ and bounded for each φ. Moreover $L_-(e^{i\varphi})$ has an analytic operator valued extension to $|z| > 1$ which is invertible for these z, $L_+(e^{i\varphi})$ has an analytic operator valued extension to $|z| < 1$ invertible for these z. This factorization is unique if $L_-(\infty) = I$.

(C) T_g is invertible.

Proof. Assume (A). Consider

$$\mu B(e^{i\varphi}) + C(e^{i\varphi}) \tag{3.1}$$

where first $\mu = A(e^{i\theta_0})$ for some real θ_0. Conditions (1) and (3) and the results of [1] guarantee that a factorization

$$\mu B(e^{i\varphi}) + C(e^{i\varphi}) = D_-(\mu, e^{i\varphi})D_+(\mu, e^{i\varphi}) \tag{3.2}$$

exists for each such μ where the factors D_- and D_+ have the same properties as functions as $L_-(e^{i\varphi})$ and $L_+(e^{i\varphi})$ have as operators. Since property 3 is a homotopic invariant, such a factorization fails to exist for some μ in the spectrum of A if and only if $\exists \mu_0$ in the spectrum of A and some real φ_0 with

$$\mu_0 B(e^{i\varphi_0}) + C(e^{i\varphi_0}) = 0 . \tag{3.3}$$

If $B(e^{i\varphi_0}) \neq 0$, then $C(e^{i\varphi_0}) \neq 0$ by condition (1). Thus,

$$\mu_0 = -\frac{C(e^{i\varphi_0})}{B(e^{i\varphi_0})} . \tag{3.4}$$

But by condition (2) the change in argument as θ goes from 0 to 2π of $[a(e^{i\theta}) - \mu_0] = 0$, thus μ_0 does not belong to the spectrum of A.

Thus the factorization (3.4) exists for all μ in the spectrum of A. It is clear from the construction involved in [1] that the factors are locally analytic in μ for μ in the spectrum of A.

We normalize so that $D_-(\mu, \infty) = 1$. Then the equation (for any μ in the spectrum of A)

$$P_+^{(\varphi)}[\mu B(e^{i\varphi}) + C(e^{i\varphi})]h_\mu(e^{i\varphi}) = 1 , \tag{3.5}$$

on the space of Fourier transforms of semi-infinite sequences with one subscript, has the unique solution

$$h_\mu(e^{i\varphi}) = D_+^{-1}(\mu, e^{i\varphi}) .$$

Single-valuedness of the factors is now immediate. Thus each factor is analytic separately in μ on the spectrum of A. Moreover, for such μ, $D_-(\mu, e^{i\varphi})$ has an analytic extension to $|z| > 1$, invertible for $|z| \geqq 1$. Thus, the operator $D_-(A, e^{i\varphi})$ has the same properties, by the spectral mapping theorem. We may make the analogous statements about $D_+(\mu, e^{i\varphi})$.

Thus, by the spectral mapping theorem we may replace μ by A in $D_-(\mu, e^{i\varphi})$, $D_+(\mu, e^{i\varphi})$ and obtain $L_-(e^{i\varphi})$, $L_+(e^{i\varphi})$ with all the appropriate properties of analyticity in z and invertibility.

Next, suppose

$$M_-(e^{i\varphi})M_+(e^{i\varphi}) = L_-(e^{i\varphi})L_+(e^{i\varphi}), \tag{3.6}$$

M_-, M_+ having the same properties, then

$$L_-^{-1}(e^{i\varphi})M_-(e^{i\varphi}) = L_+(e^{i\varphi})M_+^{-1}(e^{i\varphi}) \tag{3.7}$$

and hence they are both analytic in the whole plane and equal to the identity at ∞, or

$$L_-(e^{i\varphi}) = M_-(e^{i\varphi}) , \qquad L_+(e^{i\varphi}) = M_+(e^{i\varphi}) \tag{3.8}$$

or the factorization is unique.

Thus $A \Rightarrow B$.

Now, let us assume B. We wish to solve:

$$P_+^{(\varphi)}L(e^{i\varphi})\hat{\xi} = \hat{\eta} . \tag{3.9}$$

Consider the operator $P_+^{(\varphi)}L_-^{-1}(e^{i\varphi})$.

By the isometry of the Fourier transform

$$\| P_+^{(\varphi)}L_-^{-1}(e^{i\varphi}) \| \leqq \sup_{0 \leqq \varphi \leqq 2\pi} ||| L_-^{-1}(e^{i\varphi}) ||| \tag{3.10}$$

where $|||\ \ |||$ denotes operator norm on the space of semi infinite sequences with one subscript. Similarly

$$\| L_+^{-1}(e^{i\varphi}) \| \leqq \sup_{0 \leqq \varphi \leqq 2\pi} ||| L_+^{-1}(e^{i\varphi}) ||| . \tag{3.11}$$

Let

$$\hat{\xi} = L_+^{-1}(e^{i\varphi})P_+^{(\varphi)}L_-^{-1}(e^{i\varphi})\hat{\eta} .$$

Then

$$P_+L(e^{i\varphi})\hat{\xi} = \hat{\eta} + P_+^{(\varphi)}L_-(e^{i\varphi})(P_+^{(\varphi)} - I)L^{-1}(e^{i\varphi})\hat{\eta} ,$$

but by the anti-analyticity of $L_-(e^{i\varphi})$ and the definition of $P+^{(\varphi)}$ we have

$$P_+^{(\varphi)}L(e^{i\varphi})\hat{\xi} = \hat{\eta} .$$

Thus T_g has a right inverse. This right inverse is easily shown to be a left inverse using the anti-analyticity of $L_-^{-1}(e^{i\varphi})$.

Next we assume C. Suppose $G(e^{i\theta_0}, e^{i\varphi_0}) = 0$.

Then, if T_g is invertible, so is $T_{g_M} + \delta I$, for all M large enough and all $|\delta|$ small enough and:

$$G_M(e^{i\theta}, e^{i\varphi}) = \sum_{j=-M}^{M} a_j e^{-ij\theta} \sum_{k=-M}^{M} b_k e^{-ik\varphi} + \sum_{k=-M}^{M} b_k e^{-ik\varphi} \tag{3.12}$$

Moreover, we may choose M_0, φ_1, θ_1, δ_0 such that $|\delta_0| < \delta$ and $M_0 \geqq M$ and

$$G_{M_0}(e^{i\theta_1}, e^{i\varphi_1}) + \delta_0 = 0 . \tag{3.13}$$

Next consider the sequence of vectors ξ^N, where

$$\begin{aligned} \xi^N_{jk} &= \frac{1}{\sqrt{N+1}} e^{i(j\theta_1 + k\varphi_1)} \text{ if } 0 \leqq j,\ k \leqq N \\ \xi^N_{jk} &= 0 \text{ otherwise} . \end{aligned} \tag{3.14}$$

Clearly

$$\| \xi^N \| = 1 \text{ while } \lim_{N\to\infty} (T_{g_{M_0}} + \delta_0 I)\xi^N = 0 .$$

Contradiction.

Now suppose the change in argument in condition 2 is $2\pi\eta_\theta \neq 0$. (This number is obviously independent of φ). Thus, for M large, $G_M(e^{i\theta}, e^{i\varphi})$ has the same η_θ for each φ_0. If $\eta_\theta < 0$, then $L_M(e^{i\varphi_0})$ annihilates some vector

$$K = \{k_0, k_1, \cdots\} , \qquad |||K||| = 1 .$$

But then the sequence

$$\begin{aligned} \xi^N_{kj} &= \frac{1}{\sqrt{N+1}} k_i e^{ij\varphi_0} \text{ if } 0 \leqq j \leqq N , \\ \xi^N_{ij} &= 0 \text{ if } N < j, \text{ has the property} \end{aligned} \tag{3.15}$$

$$\| \xi^N \| = 1, \text{ but } \lim_{N\to\infty} T_{g_M}\xi^N = 0 . \tag{3.16}$$

Thus T_{g_M} is not invertible, hence neither is T_g. If $\eta_\theta > 0$, we merely consider T_g^*. Finally, we assume that the change in argument is $2\pi\ \eta_\varphi \neq 0$ in condition (3). Then consider

$$e^{-i\eta_\varphi\varphi}G(e^{i\theta}, e^{i\varphi}) = H(e^{i\theta}, e^{i\varphi}) .$$

This function obeys the conditions of (A), hence it is factorable and T_h is invertible. However, if $\eta_\varphi > 0$, then

$$S^{*\eta_\varphi} = S_\varphi^{*\eta_\varphi} T_g(T_g)^{-1} = T_h(T_g)^{-1} \tag{3.17}$$

where S_φ is the right shift operator on the j subscript. This is impossible since the two operators on the right are invertible. If $\eta_\varphi < 0$, we merely consider the adjoint.

Thus (C) $\Rightarrow$ (A), and we are finished.

4. Example.

Let $B(e^{i\varphi})$ and $C(e^{i\varphi})$ have finite expansions

$$\begin{aligned} B &= \sum_{j=-N}^{M} b_j e^{-ij\varphi} \\ C &= \sum_{j=-N}^{M} c_j e^{-ij\varphi} \end{aligned} \tag{4.1}$$

and suppose

$$\mu b_{-N} + c_{-N} \neq 0 \quad \text{for} \quad \mu \text{ in the spectrum of } A . \tag{4.2}$$

Assume conditions 1, 2, 3. Then we may factor

$$\mu B(z) + C(z) = (\mu b_{-N} + c_{-N}) \prod_{i=1}^{N} (z - x_i(\mu)) \prod_{j=1}^{M} \left(1 - \frac{y_j^{(\mu)}}{z}\right) \tag{4.3}$$

for all μ in the spectrum of A and each $|x_i(\mu)| > 1$, $|y_j(\mu)| < 1$. See [3]. Then it follows that

$$L(z) = L_-(z)L_+(z) \tag{4.4}$$

with

$$L_-(z) = \prod_{j=1}^{M} \left(1 - \frac{y_j^M(A)}{z}\right) \tag{4.5}$$

$$L_+(z) = (Ab_{-N} + c_{-N}) \prod_{i=1}^{N} (z - x_i(A)) . \tag{4.6}$$

We expect this factorization to play an important role in the study of difference equations arising from hyperbolic systems in regions in space having corners.

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