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APPROXIMATION OF WIENER INTEGRALS OF FUNCTIONALS CONTINUOUS IN THE UNIFORM TOPOLOGY

H. C. FINLAYSON

The result obtained in this paper is a technique for the approximation and estimation of error of Wiener integrals of suitable functionals continuous in the uniform topology. For a certain class of functionals called third degree polynomials exact results occur at the first as well as each subsequent stage of approximation.

Similar results for functionals continuous in the Hilbert topology are given in [1], [4], [5], [6] and [7]. In each of these papers the functions $x(s)$ of Wiener space are approximated by linear combinations of the first n indefinite integrals $\{\beta_i(t)\}$ of a certain complete set of orthonormal functions $\{\alpha_i(t)\}$. The approximation for $x(t)$ turns out to be $\sum_{i=1}^{n} c_i(x)\beta_i(t)$ where the $c_i(x)$'s are Stieltjes integrals of $x(s)$ with respect to the α's. When $x(t)$ is replaced by this approximation in $F[x(\cdot)]$ a standard Wiener integration formula can be applied. If F is required to be continuous in the Hilbert topology, [4] and [5] show there is (as might be expected) considerable choice in the C.O.N. set. However the uniform topology seems more natural to use in Wiener space and when continuity in this topology is required it may be there is not so large a choice. The Haar functions seem a reasonable choice to try and it is these the author has used.

Let C be the space of real functions continuous on [0, 1] and which vanish at zero. Let $\{h_n(s)\}$ be the Haar functions normalized to be right continuous and to vanish at $s = 1$. The approximation of this paper applies to $F[x]$ if

$$F[x_0 + x] = F[x_0] + \sum_{i=1}^{3} \int_0^1 (i) \int_0^1 x(s_1) \cdots x(s_i) d_{(i)} K_i(x_0 | s_1, \cdots, s_i) + Q[x_0, x]$$

where, with $||x|| = \max_t |x(t)|$, $|Q[x_0, x]| \leqq A||x||^D \exp B(||x_0||^2 + ||x||^2)$ with $B < 1/12$ and $D > 0$.

Notation. Let $\{h_n(s)\}$ be the Haar functions on [0, 1] normalized to be right continuous and so $h_n(1) = 0$.

Let, for $n = 1, 2, 3, \cdots$,

$$c_n(x) = -\int_0^1 x(s) dh_n(s) ,$$

$$\beta_n(t) = \int_0^t h_n(s)ds \,,$$

$$x^n(t) = \sum_{i=1}^n c_i(x)\beta_i(t) \,,$$

$$\psi_n(\xi, t) = \sum_{i=1}^n \xi_i\beta_i(t) \,,$$

$$e_n(\xi) = (2\pi)^{-n/2} \exp [(-\xi_1^2 - \cdots - \xi_n^2)/2] \,.$$

(This is the kernel commonly used now whereas that used in [1], [4] and [5] was $\pi^{-n/2} \exp (-\xi_1^2 - \cdots - \xi_n^2)$.)

Finally let

$$||x|| = \sup_{t \in [0,1]} |x(t)|$$

for $x \in C$ and let

$$\rho(s, t) = (2^{3/2}/\pi)\sum_{k=1}^\infty \sin (k - \tfrac{1}{2})\pi t h_k(s)/(2k - 1)$$

(that this last series converges for $(s, t) \in [0, 1] \times [0, 1]$ and is, for fixed s, continuous in t will be seen in Theorem 1. Also $\rho(s, 0) = 0$ and so, for fixed s, $\rho(s, t)$ is in C and $\rho^n(s, t)$ can be computed).

In connection with Radon integrals the symbol $\int_0^1$ will be used rather than $\int_0^1 (n) \int_0^1$ and d subscripted with n subscripted s's will be replaced by $d_{(n)}$. Another abbreviation is given by the following equation:

$$\int_{-\infty}^\infty G(f(\xi), n)d\mu_m = \int_{-\infty}^\infty (n) \int_{-\infty}^\infty e_n(\xi)G(f(\xi), n)d\xi_i \cdots d\xi_n \,.$$

If $F[x]$ is defined on C we define $I_n(F)$ and $J_n(F)$ by the following equations provided the right hand sides have meaning.

$$I_n(F) = \int_{-\infty}^\infty F[\psi_n(\xi, \cdot)]d\mu_n \,,$$

$$J_n(F) = \tfrac{1}{2}\int_{-\infty}^\infty \int_0^1 \{F[\psi_n(\xi, \cdot) + \rho(s, \cdot) - \rho^n(s, \cdot)] + F[\psi_n(\xi, \cdot) - \rho(s, \cdot) + \rho^n(s, \cdot)]\}dsd\mu_n \,.$$

2. The principal theorem. The following theorem and corollary are the main results of the paper.

THEOREM 6. *Let $F[x]$ be integrable on C and such that $J_n(F)$: $n = 1, 2, 3, \cdots$ exists as a finite quantity. For each $x_0 \in C$ let:*

$K_i(x_0 | s_1, \cdots, s_i)$, $i = 1, 2, 3$, be right continuous and of bounded variation in any $j (j \leqq i)$ of the variables for the other $i - j$ variables fixed. For each pair $[x_0, x] \in C \times C$

$$let\ P[x_0, x] = F[x_0] + \sum_{i=1}^{3} \int_0^1 x(s_1) \cdots x(s_i) d_{(i)} K(x_0 | s_1, \cdots, s_i) .$$

Let $F[x_0 + x] = P[x_0, x] + Q[x_0, x]$ *define* $Q[x_0, x]$. *Then if* $|Q[x_0, x]| \leqq A\|x\|^D \exp B[\|x_0\|^2 + \|x\|^2]$, *where* $B < 1/12$, *and* $D > 0$ *and if* $\alpha \in (0, 1/2)$ *there follows*

$$\left| J_n(F) - \int_C F[x] dx \right| = o(n^{-\alpha D}) \text{ as } n \longrightarrow \infty .$$

Furthermore, if $Q = 0$ *then*

$$J_n(F) = \int_C F[x] dx \text{ for each } n .$$

COROLLARY. *Under the conditions of the above theorem a specific estimate of error is given by*

$$\begin{aligned} &\left| J_n(F) - \int_C F[x] dx \right| \\ &\quad \leqq A\{M^{2/3}[2^{\alpha+1}/n^\alpha]^D [2/\sqrt{1 - 12B}]^{1/3} \\ &\quad + [2/\sqrt{1 - 4B}][31/\sqrt{n}]^D \exp [31^2 B/n] \end{aligned}$$

where M *is the constant given in Lemma* 2 *with* P *replaced by* $3D/2$.

The following theorems (except Theorem 2) and two lemmas are the main results used in the proof of Theorem 6. These theorems are analogous to correspondingly numbered theorems in [4]. In fact Theorems 3 and 5 are identical to those of [4] and so proofs for them will not be given.

THEOREM 1. (i) *The* $\rho(s, t)$ *series converges, the convergence being uniform in* $(s, t) \in [0, 1] \times [0, 1]$.

(ii) $\rho(s, t)$ *is continuous in* t *for each fixed* s.

(iii) $\|\rho(s, \cdot) - \rho^n(s, \cdot)\|$ *is measurable in* s.

(iv) $\|\rho(s, \cdot) - \rho^n(s, \cdot)\| \leqq 31/n^{1/2}$.

(v) *If, for* $x \in C$, $F[x] = K_0 + \sum_{i=1}^{3} \int_0^1 x(s_1) \cdots x(s_i) d_{(i)} K_i(s_1, \cdots, s_i)$ *in which the* K_i*'s are right continuous and of bounded variation in any* $j (j \leqq i)$ *of the variables for the other* $i - j$ *variables fixed then*

$$\int_C F[x] dx = \frac{1}{2} \int_0^1 \{F[\rho(s, \cdot)] + F[-\rho(s, \cdot)]\} ds . \tag{2.1}$$

(The reason $\sqrt{2}$ does not appear under the ρ's as in Theorem 1 of [4] is the change in kernel which results in $\int_C x(s)x(t)dx$ being $\frac{1}{2} \min (s, t)$ rather than $\min (s, t)$.)

LEMMA 1. (*Ciesielski* [2]). *For each $x \in C$, the graph of $x^n(t)$ is an inscribed polygon of the graph of $x(t)$. The graph of $x^{n+1}(t)$ has at least the same vertices as that of $x^n(t)$ and $\{x^n(t)\}$ converges uniformly to $x(t)$.*

Some notation, now to be given, is used in Lemma 2 below. For fixed $x \in C$ and $\alpha \in (0, \frac{1}{2})$ let $\varphi_\alpha[x]$ be the infimum of $h > 0$ such that $|x(t') - x(t'')| \leqq h|t' - t''|^\alpha$ for t' and t'' in [0, 1]. (that such h exists for almost all $x \in C$ has been shown by N. Wiener [10]).

LEMMA 2. (*Yeh* [8]). *For every $\alpha \in (0, 1/2)$ and $P > 0$, the functional $\{\varphi_\alpha[x]\}^P$ is Wiener integrable i.e., $\int_C \{\varphi_\alpha[x]\}^P dx < \infty$.*

In fact for any $N > \frac{1}{2} \max\{(1 + 2\alpha)/(1 - 2\alpha), P\}$,

$$\int_C \{\varphi_\alpha[x]\}^P dx \leqq M$$

where

$$M = (2N)^N e^{-N}\{1 - 2^{1/2+\alpha-N(1-2\alpha)}\}^{-1}\sum_{m=1}^{\infty}(m + 1)^P/(2N + 1) < \infty .$$

THEOREM 2. *If $F[x]$ is continuous in the uniform topology on C and if either*

(i) *$F[x]$ is bounded*

or

(ii) *there exist nondecreasing $G_1(u)$ and $G_2(u)$ defined on $[0, \infty)$ such that $G_1[\max_{t\in[0,1]} x(t)]$ and $G_2[\max_{t\in[0,1]}\{-x(t)\}]$ are Wiener integrable and such that*

$$|F[x] \leqq G_1[\max_{t\in[0,1]} x(t)] + G_2[\max_{t\in[0,1]}\{-x(t)\}]$$

then

$$\lim_{n\to\infty} I_n(F) = \int_C F[x]dx . \tag{2.2}$$

Particular suitable choices for G_1 and G_2 are

(2.3) $G_1(u) = G_2(u) = M \exp\{hu^p\}$ *for $p \in [0, 2)$ and arbitrary real M and h.*

(2.4) $G_1(u) = G_2(u) = M \exp\{hu^2\}$ *for $h < \frac{1}{2}$ and arbitrary real M.*

THEOREM 3. *If $F[x] \in L_1(C)$ then*

$$\int_C F[x]dx = \int_{-\infty}^{\infty}\int_C F[x(\cdot) - x^n(\cdot) + \psi_n(\xi, \cdot)]dx d\mu_n .$$

THEOREM 4. *For $\alpha \in (0, 1/2)$ and $P \geqq 0$,*

$$\int_C \|x - x^n\|^P dx \leqq M[2^{\alpha+1}/n^\alpha]^P$$

where M is as in Lemma 2.

THEOREM 5. *For fixed $i \in \{1, 2, 3\}$, let $H(t_1, \cdots, t_i)$ be right continuous and of bounded variation in any $j (j \leqq i)$ of its variables for the other $i - j$ variables fixed. Then there exists $N(s_1, \cdots, s_i)$ of bounded variation and right continuous such that for all $x \in C$,*

$$\int_0^1 [x(t_1) - x^n(t_1)] \cdots [x(t_i) - x^n(t_i)] d_{(i)} H(t_1, \cdots, t_i)$$

is of the form

$$\int_0^1 x(s_1) \cdots x(s_i) d_{(i)} N(s_1, \cdots, s_i) .$$

The proof of Theorem 6 and its corollary follows. Let

$$\begin{aligned}\varepsilon_n &= \int_C F[x]dx - J_n(F) \\ &= \int_C F[x]dx \\ &\quad - \int_{-\infty}^{\infty} \tfrac{1}{2}\int_0^1 \{F[\psi_n(\xi, \cdot) + \rho(s, \cdot) - \rho^n(s, \cdot)] \\ &\quad + F[\psi_n(\xi, \cdot) - \rho(s, \cdot) + \rho^n(s, \cdot)]\} ds d\mu_n .\end{aligned}$$

If now F is replaced by $P + Q$ the integrals can be combined and, because of Theorems 1, 3 and 5, the part involving P disappears. (The detailed argument is exactly the same as that in [4, pp. 64–65] where all symbols and theorems used there are to be replaced by the corresponding ones of this paper. See also the note after (2.1)). What is left is

$$\begin{aligned}\varepsilon_n = \int_{-\infty}^{\infty} \Big\{ &\int_C Q[\psi_n(\xi, \cdot), x(\cdot) - x^n(\cdot)]dx \\ &- \tfrac{1}{2}\int_0^1 (Q[\psi_n(\xi, \cdot), \rho(s, \cdot) - \rho^n(s, \cdot)] \\ &+ Q[\psi_n(\xi, \cdot), -\rho(s, \cdot) + \rho^n(s, \cdot)] ds \Big\} d\mu_n .\end{aligned}$$

If

$$|Q[x_0, x]| \leqq A\|x\|^D \exp B[\|x_0\|^2 + \|x\|^2]$$

then

$$
\begin{aligned}
|\varepsilon_n| \leqq A\int_{-\infty}^{\infty}\Big\{\int_c \|x - x^n\|^D \exp B[\|\psi_n(\xi, \cdot)\|^2 + \|x - x^n\|^2]dx \\
+ \int_0^1 \|\rho(s, \cdot) - \rho^n(s, \cdot)\|^D \exp B[\|\psi_n(\xi, \cdot)\|^2 \\
+ \|\rho(s, \cdot) - \rho^n(s, \cdot)\|^2]ds\Big\}d\mu_n .
\end{aligned}
$$

Now steps almost identical to those of [4, pp. 66–67] with $\|x\|^2$ replacing $\int_0^1 [x(s)]^2 ds$ and $a^2 + b^2 = [(a+b)^2 + (a-b)^2]/2$ replaced by

$$\|x\|^2 + \|y\|^2 \leqq \|x + y\|^2 + \|x - y\|^2 \text{ yield}$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\Big\{\int_c \|x - x^n\|^D \exp B[\|\psi_n(\xi, \cdot)\|^2 + \|x - x^n\|^2]dx\Big\}d\mu_n \\
&\qquad \leqq \Big[\int_c \|x - x^n\|^{3D/2}dx\Big]^{2/3}\Big[\int_c \exp 6B\|x\|^2 dx\Big]^{1/3}
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \exp B\|\psi_n(\xi, \cdot)\|^2 d\mu_n \int_0^1 \|\rho(s, \cdot) - \rho^n(s, \cdot)\|^D \\
&\exp B\|\rho(s, \cdot) - \rho^n(s, \cdot)\|^2 ds \leqq \int_c \exp 2B\|x\|^2 dx \, . \\
&\int_0^1 \|\rho(s, \cdot) - \rho^n(s, \cdot)\|^D \exp[B\|\rho(s, \cdot) - \rho^n(s, \cdot)\|^2]ds \, .
\end{aligned}
\tag{2.6}
$$

Finally one notes that

$$\|x\| = \max\{\max_{t\in[0,1]} x(t), \max_{t\in[0,1]}[-x(t)]\}$$

so that for $K \in (0, 1)$

$$\exp(K\|x\|^2) \leqq \exp(K\{\max x(t)\}^2) + \exp(K\{\max[-x(t)]\}^2)$$

and

$$
\begin{aligned}
&\int_c \exp(K\|x\|^2)dx \leqq 2\int_c \exp(K\{\max x(t)\}^2)dx \\
&\qquad = 4\int_0^{\infty} \exp[-(1 - 2K)u^2/2]du/\sqrt{(2\pi)} = 2/\sqrt{1 - 2K} \, .
\end{aligned}
\tag{2.7}
$$

(for the distribution of $\max x(t)$ see [3]).

The estimate (2.7) used first with Theorem 4 and then with Theorem 1 (iv) provides the estimates of the right sides of (2.5) and (2.6). The estimate given in the corollary follows at once as does also the order estimate of the theorem.

3. Proof of Theorems 1, 2 and 4. As noted in §2 after the statement of Theorem 6, only Theorems 1, 2, and 4 remain to be proved. Yeh's lemma [8] and Ciesielski's lemma [2] provide a proof for Theorem 4. The lemma due to Ciesielski will be used in the proofs of Theorems 1 and 2. An outline of the proof of this lemma will follow. First there will be noted that there is a natural double indexing of the Haar functions:

$$\alpha_0^{(0)}(s)\ ,$$
$$\alpha_n^{(k)}(s): n = 0, 1, 2, \cdots, k = 1, 2, \cdots, 2^n\ .$$

A corresponding double indexing applies to the β's. It will be convenient to speak of "the nth cycle of α's (β's): $n \geqq 0$" by which will be meant $\{\alpha_n^{(k)}: k = 1, 2, \cdots, 2^n\}$ (or similar for β's). Note that $\alpha_0^{(0)}$ is not in a cycle. Now it is fairly easy to prove by induction than any partial sum of the $c\beta$-series to at least the end of the $(N-1)^{\text{th}}$ cycle gives the value of $x(t)$ for all t of the form $l/2^N$: $l = 1, 2, \cdots, 2^N$ and that the graph of this partial sum is polygonal with vertices precisely those points where the graph of the partial sum agrees with the graph of $x(t)$. The conclusions of the lemma are thus obtained.

The proof of Theorem 2 follows:

First there is noted that a functional continuous in the uniform topology is Wiener measurable. Lemma 1 together with Lebesgue's bounded or dominated convergence theorem completes the proof of (i) or (ii) respectively. That (2.3) or (2.4) provide suitable choices for the G's follows from the formula for the integral of a functional of $\max x(t)$ which yields

$$\begin{aligned}\int_C G_1[\max x(t)]dx &= \int_C G_1[\max -\{x(t)\}]dx \\ &= 2\int_0^\infty e^{-\xi^2/2}G_1(\xi)d\xi/\sqrt{(2\pi)} \\ &= 2A\int_0^\infty e^{-\xi^2/2+h\xi^p}d\xi/\sqrt{(2\pi)}\end{aligned}$$

and the last integral clearly converges for the conditions given on p and h in (2.3) and (2.4).

Next is given the proof of Theorem 1.

(i) For any fixed s there is a most one $h_k(s)$ in "the n^{th} cycle of Haar Functions" (for this notion c.f. beginning of outline of proof of the lemma) which is not zero and $|h_k(s)| \leqq \sqrt{2^n}$. But the k for that $h_k(s)$ satisfies $k \geqq 1 + 1 + 2 + 4 + \cdots + 2^{n-1} = 2^n$. A comparison of the series, after terms of value zero have been deleted, of

$$\sum_{k=1}^\infty |\sin (k - \tfrac{1}{2})\pi t h_k(s)/(2k-1)|$$

with the series

$$\sum_{n=1}^{\infty}(\sqrt{2^n}/2^n)\ ,$$

which converges, yields the conclusion of (i).

(ii) That $\rho(s, t)$ is continuous in t for each fixed s follows at once from uniform convergence of a series of continuous functions.

(iii) To show that $||\rho(s, \cdot) - \rho^n(s, \cdot)||$ is measurable in s, $\rho^n(s, t)$ will first be calculated.

$$\rho^n(s, t) = -\sum_{i=1}^{n}\int_0^1 \rho(s, u)dh_i(u)\beta_i(t)$$

(the Stieltjes integrals exist since $\rho(s, u)$ is continuous in u)

$$= (-2^{3/2}/\pi)\sum_{i=1}^{n}\int_0^1\left[\sum_{k=1}^{\infty}\sin(k-\tfrac{1}{2})\pi u h_k(s)/(2k-1)\right]dh_i(u)\beta_i(t)$$

$$= (-2^{3/2}/\pi)\sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_0^1 \sin(k-\tfrac{1}{2})\pi u dh_i(u)h_k(s)\beta_i(t)/(2k-1)$$

(because of uniform convergence of a series of continuous functions). Thus $\rho^n(s, t)$ is measurable in s for each fixed t and so of course is $\rho(s, t)$. Since $\rho^n(s, t) - \rho(s, t)$ is continuous in t, $||\rho^n(s, \cdot) - \rho(s, \cdot)||$ is determined by a countable number of t values and so is measurable in s.

(iv) That

$$||\rho(s, \cdot) - \rho^n(s, \cdot)|| \leqq 31/n^{1/2}$$

uniformly in s is seen as follows. Let k be such that

$$1 + 1 + 2 + 4 \cdots + 2^{n-1}(=2^n) \leqq k$$
$$\leqq 1 + 1 + 2 + 4 + \cdots + 2^n(=2^{n+1})\ .$$

Note that $n \leqq \log_2 k \leqq n + 1$.

Now

$$|\rho(s, t) - \rho^k(s, t)| = (2^{3/2}/\pi)\left|\sum_{i=1}^{\infty}\left[\sin(i-\tfrac{1}{2})\pi t\right.\right.$$
$$\left.\left.+ \sum_{j=1}^{k}\int_0^1 \sin(i-\tfrac{1}{2})\pi u dh_j(u)\beta_j(t)\right]h_i(s)/(2i-1)\right|\ .$$

Since $\sin(i-\tfrac{1}{2})\pi t \in C$, there follows from Lemma 1 that

$$\left\|\sum_{j=1}^{k}\int_0^1 \sin(i-\tfrac{1}{2})\pi u dh_j(u)\beta_j(\cdot)\right\| \leqq ||\sin(i-\tfrac{1}{2})\pi\cdot|| = 1 \tag{3.1}$$

for all i. Now let the series (in i) for $\rho(s, t) - \rho^k(s, t)$ be split into two parts, viz. a finite sum from $i = 1$ to $i = k$ and the remainder of the series from $i = k + 1$ onward. The second of these two parts is estimated as follows:

$$(2^{3/2}/\pi)\left|\sum_{i=k+1}^{\infty}\cdots\right| \leqq 2^{5/2}\sum_{i=n}^{\infty}(1/\sqrt{2})^i/\pi$$

(because of (3.1), the comparison series mentioned in the proof of Theorem 1(i), and the relation between k and n)

$$=2^{5/2}(1/\sqrt{2})^n/[\pi(1-1/\sqrt{2})] \leqq 32/(\pi k^{1/2})\ .$$

To estimate the first part it will be noted that, for any i, the maximum difference between the graph of $\sin(i-\frac{1}{2})\pi t$ and the k^{th} polygonal approximation, viz. $(\sin(i-\frac{1}{2})\pi t)^k$, is no greater than the maximum slope of this sine curve multiplied by $1/2^n$. Thus

$$\left\|\sin(i-\tfrac{1}{2})\pi\cdot+\sum_{j=1}^{k}\int_0^1\sin(i-\tfrac{1}{2})\pi u dh_j(u)\beta_j(\cdot)\right\| \leqq (i-\tfrac{1}{2})\pi/2^n \leqq i\pi/2^n$$

and therefore

$$(2^{3/2}/\pi)\left|\sum_{i=1}^{k}\cdots\right| \leqq (2^{3/2}/\pi)\left[\pi/2^n+\sum_{j=0}^{n}(2^{j+1}\pi/2^n)(1/\sqrt{2})^j\right]$$

(because, for given s, the one function in the j^{th} cycle of Haar functions which is not zero has index no greater than 2^{j+1}: the $\pi/2^n$ before the summation is due to $\alpha_0^{(0)}$ which is not in a cycle)

$$=(2^{3/2}/2^n)\left[1+2\sum_{j=0}^{n}\sqrt{2^j}\right] \leqq 20/k^{1/2}\ ,$$

and addition of the estimates completes the proof.

To prove (v) there is noted that the Fubini theorem for mixed Stieltjes and Wiener integrals will yield the required result if (2.1) can be shown to hold for $F[x]$ any one of the forms K_0, $x(s_1)$, $x(s_1)x(s_2)$ and $x(s_1)x(s_2)x(s_3)$. But (2.1) clearly does hold for K_0 (yielding K_0) and for $x(s_1)$ and $x(s_1)x(s_2)x(s_3)$ (yielding 0). That (2.1) holds for $x(s_1)x(s_2)$ is seen from the computation

$$\int_0^1\rho(u,s_1)\rho(u,s_2)du$$

$$=(2^3/\pi^2)\sum_{k=1}^{\infty}\sin(k-\tfrac{1}{2})\pi s_1\sin(k-\tfrac{1}{2})\pi s_2/(2k-1)^2=\min(s_1,s_2)$$

(by Mercer's theorem for the integral equation

$$\phi_n(s)=\lambda_n\int_0^1\min(s,t)\phi_n(t)dt\ .$$

[9, p. 136] or [7, p. 464]) and the proof is complete.

Finally there follows the proof of Theorem 4. Let k be such that

$$1 + 1 + 2 + 4 + \cdots + 2^{n-1}(=2^n) \leqq k$$
$$\leqq 1 + 1 + 2 + 4 + \cdots + 2^n(=2^{n+1})$$

and let $t \in [0, 1]$ be such that

$$r/2^n \leqq t \leqq (r+1)2^n : 0 \leqq r \leqq 2^n - 1 .$$

Now (see notation in Yeh's lemma) for almost all $x \in C$

$$|x(t) - x(r/2^n)| \leqq \phi_\alpha[x]/2^{\alpha n}. \quad \text{Also} \tag{3.2}$$

$$|x^k(r/2^n) - x^k(t)| \leqq |x^k(r/2^n) - x^k([r+1]/2^n)| \tag{3.3}$$

(because the graph of x^k is a chord of the graph of x on $[r/2^n, (r+1)/2^n]$ according to the Ciesielski lemma).

$$= |x(r/2^n) - x([r+1]/2^n)|$$

(since, from the Ciesielski lemma, x and x^k agree at $r/2^n$ and $[r+1]/2^n$)

$$\leqq \phi_\alpha[x]/2^{\alpha n}$$

for almost all x.

Thus

$$\begin{aligned} &|x(t) - x^k(t)| \\ &\quad = |x(t) - x(r/2^n) + x(r/2^n) - x^k(t)| \\ &\quad = |x(t) - x(r/2^n) + x^k(r/2^n) - x^k(t)| \\ &\quad \leqq 2\phi_\alpha[x]/2^{\alpha n} \end{aligned}$$

(because of the Schwarz inequality and inequalities (3.2) and (3.3)). From the fact that $n \geqq \log_2 k - 1$ there then follows for almost all x

$$||x - x^k|| \leqq 2^{\alpha+1}\phi_\alpha[x]/k^\alpha$$

and an application of Yeh's lemma completes the proof.

References

1. R. H. Cameron, *A "Simpson's Rule" for the numerical evaluation of Wiener's integrals in function space*, Duke Math. J. **18** (1951), 111-130.

2. Z. Ciesielski, *On Haar functions and on the Schauder basis of the space* $C_{\langle 0,1\rangle}$, (Russian summary, unbound insert), Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **7** (1959), 227-232.

3. P. Erdos & M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. **52** (1946), 292-302.

4. H. C. Finlayson, *New approximations for Wiener integrals, with error estimates*, Canad. J. Math. **19** (1967), 58-105.

5. ———, *Vladimirov-type approximation of Wiener integrals of suitably dominated functionals*, Trans. Amer. Math. Soc. **132** (1968), 461-469.

6. Alan G. Konheim and Willard L. Miranker, *Mathematics of computation*, **21** (1967), 49-65.
7. V. S. Vladimirov, *The approximate evaluation of Wiener integrals*, Uspehi Mat. Nauk **15** (1960), 129-135; English transl. Amer. Math. Soc. Trans. (2) **34** (1963), 405-412.
8. J. Yeh, *Minimal coefficients in Hölder conditions in the Wiener space*. (submitted for publication).
9. K. Yosheda, *Lectures on differential and integral equations*, New York, 1960.
10. N. Wiener, *Generalized harmonic analysis*, Acta Mathematica **55** (1930), 117-258.

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UNIVERSITY OF MANITOBA