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# CHARACTERIZATION OF SEPARABLE IDEALS

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**A $k$-algebra $A$ is called separable if the exact sequence of left $A^e = A \otimes_k A^0$-modules: $0 \to J \to A^e \xrightarrow{\phi} A \to 0$ splits, where $\phi(a \otimes b^0) = a \cdot b$; a two-sided ideal $\mathfrak{A}$ of $A$ is separable in case the $k$-algebra $A/\mathfrak{A}$ is separable.**

**In this note, we present two characterizations of separable ideals. In particular, one finds that a monic polynomial $f \in k[x]$ generates a separable ideal if, and only if, $f = g_1 \cdots g_s$, where the $g_i$ are monic polynomials which generate pairwise comaximal indecomposable ideals in $k[x]$, and $f'(a)$ is a unit in $k[a] = k[x]/f \cdot k[x]$ $(a = x + f \cdot k[x])$.**

Throughout this paper, we assume that all rings have units and all ring morphisms preserve units, further, all modules will be assumed unitary. We will denote the center of the ring $A$ by $Z(A)$. Each $k$-algebra $A$ induces an exact sequence of left $A^e = A \otimes_k A^0$-modules:

$$(1) \qquad 0 \longrightarrow J \longrightarrow A^e \xrightarrow{\phi} A \longrightarrow 0$$

where $\phi(a \otimes b^0) = a \cdot b$.

DEFINITION 2 [1]. A will be called a *separable* $k$-algebra if the sequence (1) splits. More generally, a two-sided ideal $\mathfrak{A}$ in the $k$-algebra $A$ will be called a *separable ideal* if the quotient algebra $A/\mathfrak{A}$ $(k \to A \to A/\mathfrak{A})$ is separable. Denote by $\mathrm{Sep}_k(A)$ the set of all such ideals in $A$; of particular interest is the subset $\mathrm{Sep}_k^*(A)$ of all separable ideals $\mathfrak{A}$ for which $A/\mathfrak{A}$ is a projective $k$-module.

PROPOSITION 3 [6]. *Let $A$ be a $k$-algebra.*

(a) $\mathfrak{A} \in \mathrm{Sep}_k(A) \wedge \mathfrak{A} \leqq \mathfrak{A}' \Rightarrow \mathfrak{A}' \in \mathrm{Sep}_k(A)$ ($\mathfrak{A}'$ *is any two-sided ideal of* $A$).

(b) *If* $(\mathfrak{A}_i)_{i=1}^n \subset \mathrm{Sep}_k(A)$ *is a family of pairwise comaximal ideals, then* $\bigcap_{i=1}^n \mathfrak{A}_i \in \mathrm{Sep}_k(A)$.

The following result found in [1] provides a criterion for answering the question, is $\mathrm{Sep}_k(A) = \varnothing$ or $\mathrm{Sep}_k(A) \neq \varnothing$.

PROPOSITION 4. *Let $A$ be a $k$-algebra, and let $K$ be a commutative $k$-algebra. If $\phi(0\colon J) \otimes_k K$ generates $Z(A) \otimes_k K$ as an ideal, then $A \otimes_k K$ is a separable $K$-algebra.*

COROLLARY 5. (a) *If $\alpha < k$ is an ideal such that*

$$\alpha \cdot Z(A) + \phi(0: J) = Z(A) \,,$$

*then* $\alpha A \in \mathrm{Sep}_k(A)$.

(b) *If* $Z(A) = k$, *and either* $\phi(0: J)$ *is not nil or* $\phi(0: J) \not\leq \mathrm{Rad}(k)$, *then* $\mathrm{Sep}_k(A) \neq \emptyset$, *where* $\mathrm{Rad}(k)$ *is the Jacobson radical of* $k$.

## 1. Representation of separable ideals.

THEOREM 1.1. *Let* $A$ *be a* $k$*-algebra and* $\mathfrak{A} \in \mathrm{Sep}_k(A)$. *If the* $k$*-module* $A/\mathfrak{A}$ *if of finite type, then for each maximal ideals* $m < k$, *there is a family* $(M_i)_{i=1}^{s} \subset \mathrm{Sep}_k(A)$ *of maximal two-sided ideals such that*

$$\mathfrak{A} + (m \cdot A) = M_1 \cap \cdots \cap M_s \,. \tag{1.2}$$

*Proof.* For each maximal ideal $m < k$, the $k/m$-algebra $k/m \otimes A/\mathfrak{A}$ is separable and of finite type as a $k/m$-module, it follows from [2] Proposition 3.2 that $k/m \otimes A/\mathfrak{A} \cong (A/\mathfrak{A})/m(A/\mathfrak{A}) \cong A/(m \cdot A + \mathfrak{A}) \cong B_1 \oplus \cdots \oplus B_s$, where each $B_i$ is a simple $k/m$-algebra with $Z(B_i)$ being a separable field extension of $k/m$; in particular, each $B_i$ is a separable $k$-algebra. Denoting by $M_i$ the kernel of the mapping $A \to A/(mA + \mathfrak{A}) \to B_i$, we find that the family $(M_i)_{i=1}^{s}$ has the desired properties.

REMARK 1.3. If, in (1.1), we assume $\mathfrak{A} \in \mathrm{Sep}_k^*(A)$, it follows from (1.1) of [9], that we can drop the assumption that $A/\mathfrak{A}$ is a $k$-module of finite type.

We obtain immediately from the local criteria for separability ([2], p. 100) the following theorem.

THEOREM 1.4. *Let* $A$ *be a* $k$*-algebra with two-sided ideal* $\mathfrak{A}$ *such that the* $k$*-module* $A/\mathfrak{A}$ *is of finite type. Suppose either that* $k$ *is Noetherian or that* $A/\mathfrak{A}$ *is a projective* $k$*-module.*

*If, for each maximal ideal* $m < k$, $\mathfrak{A} + m \cdot A$ *has a representation* (1.2) *with separable maximal ideals, then* $\mathfrak{A} \in \mathrm{Sep}_k(A)$.

COROLLARY 1.5. *Let* $k$ *be a field.*

(a) $\mathfrak{A}$ *is a separable maximal ideal of* $A$ *if, and only if,* $A/\mathfrak{A}$ *is a simple* $k$*-algebra whose center is a separable field extension of* $k$.

(b) $\mathfrak{A} \in \mathrm{Sep}_k(A)$ *if, and only if,* $\mathfrak{A}$ *is the intersection of a finite family of separable maximal ideals of* $A$.

REMARKS 1.6. (1.5) generalizes a result of [6] where a different definition of $\mathfrak{A} \in \mathrm{Sep}_k(A)$ is given. (1.5) also leads to the following fact. For a field $k$, $f \in k[x]$ generates a separable ideal if, and only if, $f'(a)$ is a unit in $k[a] = k[x]/f \cdot k[x]$, $a = x + fk[x]$, and $f$ is the product of distinct polynomials of $k[x]$.

DEFINITION 1.7. [5]. A monic polynomial $f \in k[x]$ is separable if the ideal $fk[x]$ is separable.

PROPOSITION 1.8. *If $f \in k[x]$ is separable, then $f'(a)$ is a unit in $k[a]: = k[x]/f \cdot k[x]$.*

*Proof.* Assume, first, that $k$ is local with maximal ideal $m$. Denote by $\bar{f}$ the reduction of $f$ modulo $m$, then

$$k[x]/(m, f) \simeq k/m \otimes_k k[a] \simeq k/m[x]/\bar{f}k/m[x] = k/m[\bar{a}]$$

is a separable $k/m$-algebra, hence $\bar{f}$ is a separable polynomial. Whence, by (1.6), $\bar{f} = \bar{g}_1 \cdots \bar{g}_s$ in $k/m[x]$, where each $\bar{g}_i$ is irreducible and $\bar{f}'(\bar{a})$ is a unit in $k/m[\bar{a}]$.

Now suppose $f'(a)$ is a nonunit in $k[a]$; by [7], p. 29, Lemma 4, each maximal ideal of $k[a]$ has the form $(g_i(a), m)$, where $g_i \in k[x]$ has reduction $\bar{g}_i$ modulo $m$. Thus, $f'(a) \in (g_i(a), m)$ for some $i \in [1, s]$, and this implies $f'(x) \in (g_i(x), m)$. But then

$$f'(x) \in \ker\,(k[x] \to k[x]/fk[x] \to k[x]/(m, fk[x]) \to k[x]/(g_ik[x], m))\ ,$$

so that $\bar{f}'(a)$ could not be a unit in $k/m[\bar{a}]$. This contradiction establishes our claim that $f'(a)$ is a unit when $k$ is local.

In general, observe that $f'(a)$ is a unit in $k[a]$ if, and only if $f'_m(a_m)$ is a unit in $k_m[a_m] = k_m \otimes_k k[a]$ for each maximal ideal $m < k$, and then apply the foregoing result.

PROPOSITION 1.9. *Let $f \in k[x]$ be a monic polynomial satisfying the conditions.*

( i ) *$f'(a)$ is a unit in $k[a] = k[x]/fk[x]$;*

( ii ) *$f(x) = f_1 \cdots f_s$ in $k[x]$, where the monic polynomials $f_j$ generate indecomposable ideals which are pairwise comaximal.*

*Then $f$ is separable.*

*Proof.* Let $m < k$ be a maximal ideal of $k$, and denote by $\bar{f}$ the reduction of $f$ modulo $m$, then $\bar{f}'(\bar{a})$ is a unit in $k/m[\bar{a}] = k/m \otimes_k k[a]$. Since $\bar{f} = \bar{f}_1 \cdots \bar{f}_s$ in $k/m[x]$, we see that $\bar{f}'_i = 0$ in $k/m[x]$ entails $\bar{f}' = \bar{f}'_i g + \bar{f}_i g' \in \bar{f}_i k/m[x]$. But this implies $\bar{f}'(\bar{a})$ in a nonunit in $k/m[\bar{a}]: = k/m[x]/\bar{f}k/m[x]$, since $\bar{f}_i k/m[x] < k/m[x]$. Thus, each of the $\bar{f}_i$ separable polynomials in $k/m[x]$ which generate pairwise comaximal by (ii). An application of [5] (2.3) shows that $f$ is a separable polynomial in $k[x]$.

COROLLARY 1.10. *Suppose $k$ has no proper idempotents and $f \in k[x]$ is monic. A necessary and sufficient condition that $f$ be separable is that conditions* (i) *and* (ii) *of* (1.9) *holds.*

*Proof.* We need only verify that when $f$ is separable, $f = f_1 \cdots f_s$ where each of the ideals $f_i k[x]$ is indecomposable and they are pairwise comaximal. But $k[x]/f \cdot k[x]$ has only a finite number of idempotents, since it is a free $k$-module of rank equal $\deg(f)$; hence $k[x]/f \cdot k[x] = B_1 \pi \cdots \pi B_s$, where each $B_i$ is connected and separable as well as projective as a $k$-module. Then, by [5] (2.9), $B_i = k[x]/f_i k[x]$ and we see that $f = f_1 \cdots f_s$ as usual.

## 2. Another representation of separable ideals.

DEFINITION 2.1. Let $A$ be a $k$-algebra. The two-sided ideal $\mathfrak{A} < A$ will be called *decomposable* if $\mathfrak{A} = \mathfrak{A}_1 \cap \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are proper two-sided comaximal ideals of $A$; otherwise $\mathfrak{A}$ will be called *indecomposable*. $A$ will be called decomposable or indecomposable according to whether or not 0 is.

THEOREM 2.2. *Let $k$ be a commutative ring without proper idempotents. Assume $\mathfrak{A} \in \mathrm{Sep}_k^*(A)$. Then there is a unique family $(M_i)_{i=1}^s$ of pairwise comaximal indecomposable separable ideals of $A$ such that*

$$\mathfrak{A} = M_1 \cap \cdots \cap M_s . \tag{2.3}$$

*Proof.* Since the projective $k$-module $A/\mathfrak{A}$ has finite rank, we can write $A/\mathfrak{A} \simeq B_1 \pi \cdots \pi B_s$, where the $B_i$ are indecomposable separable $k$-algebras. Putting $M_i = \ker[A \to A/\mathfrak{A} \to B_i]$ we obtain the desired family.

If $\mathfrak{A} = N_1 \cap \cdots \cap N_t$, where the $N_j$ are as the $M_i$, then $A/\mathfrak{A} = A/M_1 \pi \cdots \pi A/M_s = A/N_1 \pi \cdots \pi A/N_t$ implies that

$$1 = e_1 + \cdots + e_s = f_1 + \cdots + f_t,\ e_i,\ f_j$$

being orthogonal central idempotents. Since all the factors are indecomposable, for each $i$ there is a unique $j$ such that $f_i = f_i e_j$; hence $t \leqq S$, and by symmetry, $s \leqq t$, so $s = t$. The indecomposability also implies (after reordering) that $e_i = f_i$, so that

$$M_i = \ker[A \to (A/\mathfrak{A})e_i] = \ker[A \to (A/\mathfrak{A})f_i] = N_i ,$$

completing the proof.

REMARK 2.4. (2.2) generalizes a result obtained in [5], see p. 471, (2.10).

## BIBLIOGRAPHY

1. M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367-409.
2. H. Bass, *Lectures on topics in algebraic K-theory*, Tata Institute of Fundamental Research, Bombay 1967.
3. M. Bourbaki, *Algebre*, Chap. 8, Hermann, 1958.
4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, 1956.
5. G. J. Janusz, *Separable algebras over commutative rings*, Trans. Amer. Math. Soc. **122** (1966), 461-79.
6. R. E. MacRae, *On the motion of separable ideals*, Archiv Der Mathematik **28** (1967), 591-594.
7. J. P. Serre, *Corps locaux*, Hermann, Paris, 1962.
8. J. Sonn and H. Zassenhaus, *On the theorem on the primitive element*, Amer. Math. Monthly **74** (1967).
9. O. E. Villamayor and D. Zelinsky, *Galois theory for rings with finitely many idempotents*, Nagoya Math. J. **27** (1966), 721-731.

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