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ON THE NUMBER OF NONPIERCING POINTS IN CERTAIN CRUMPLED CUBES

ROBERT J. DAVERMAN

Let K denote the closure of the interior of a 2-sphere S topologically embedded in Euclidean 3-space E^3. If $K - S$ is an open 3-cell, McMillan has proved that K has at most one nonpiercing point. In this paper we use a more general condition restricting the complications of $K - S$ to describe the number of nonpiercing points. The condition is this: for some fixed integer n $K - S$ is the monotone union of cubes with n holes. Under this hypothesis we find that K has at most n nonpiercing points (Theorem 5). In addition, the complications of $K - S$ are induced just by these nonpiercing points. Generally, at least two such points are required, for otherwise $n = 0$ (Theorem 3).

A space K as described above is called a *crumpled cube.* The *boundary of* K, denoted $\mathrm{Bd}\, K$, is defined by $\mathrm{Bd}\, K = S$, and the *interior of* K, denoted $\mathrm{Int}\, K$, is defined by $\mathrm{Int}\, K = K - \mathrm{Bd}\, K$. We also use the symbol Bd in another sense: if M is a manifold with boundary, then $\mathrm{Bd}\, M$ denotes the boundary of M. This should not produce any confusion.

Let K be a crumpled cube and p a point in $\mathrm{Bd}\, K$. Then p is a *piercing point of* K if there exists an embedding f of K in the 3-sphere S^3 such that $f(\mathrm{Bd}\, K)$ can be pierced with a tame arc at $f(p)$.

Let U be an open subset of S^3. The *limiting genus of* U, denoted $LG(U)$, is the least nonnegative integer n such that there exists a sequence $H_1, H_2, \cdots$ of compact 3-manifolds with boundary satisfying (1) $U = \cup H_i$, (2) $H_i \subset \mathrm{Int}\, H_{i+1}$, and (3) genus $\mathrm{Bd}\, H_i = n$ $(i = 1, 2, \cdots)$. If no such integer exists, $\mathrm{LG}\,(U)$ is said to be infinite. Throughout this paper the manifolds H_i described above can be obtained with connected boundary, in which case H_i is called a *cube with n holes.*

Applications of the finite limiting genus condition are investigated in [6] and [14]. *For any crumpled cube K such that* $\mathrm{LG}(\mathrm{Int}\, K)$ is finite and $\mathrm{Bd}\, K$ is locally peripherally collared from $\mathrm{Int}\, K$, it is shown that $\mathrm{Bd}\, K$ is locally tame (from $\mathrm{Int}\, K$) except at a finite set of points. Under the hypothesis of this paper, $\mathrm{Bd}\, K$ may be wild at every point; nevertheless, with a collapsing (in the sense of Whitehead [15]) argument comparable to [13, Th. 1], the problem of counting the nonpiercing points of K is reduced to one in which the results of [6] and [14] apply.

A subset X of the boundary of a crumpled cube K is said to be *semi-cellular in* K if for each open set U containing X there exists an open set V such that $X \subset V \subset U$ and loops in $V - X$ are null homotopic in $U - X$. In the last section of this paper semi-cellular sets are discussed in order to characterize those sewings of two crumpled cubes which yield S^3, in case the limiting genus of one of the crumpled cubes is finite.

A simple closed curve J is *essential in an annulus* A if J lies in A and bounds no disk in A.

If X is a set in a topological space, then $\mathrm{Cl}\, X$ denotes the closure of X.

2. A cellularity criterion.

LEMMA 1. *Let H be a sphere with n handles. Then there exists an integer $k(n)$ such that if $J_1, \cdots, J_{k(n)}$ are mutually exclusive simple closed curves in H, no one of which bounds a disk in H, then some pair $\{J_r, J_s\}$ bounds an annulus in H.*

Proof. The number $k(n) = 2$ is known to work if $n = 1$. Otherwise, the proof proceeds by induction, using $k(n) = 3n - 2$ whenever $n \geqq 2$.

THEOREM 2. *Let C be a crumpled cube such that* $\mathrm{LG}(\mathrm{Int}\, C) = n < \infty$. *Then there exists a finite set Q of points in* $\mathrm{Bd}\, C$ *such that for each open set $U \supset \mathrm{Bd}\, C$, each point of $\mathrm{Bd}\, C - Q$ has a neighborhood V such that any loop in $V - \mathrm{Bd}\, C$ is null-homotopic in $U - \mathrm{Bd}\, C$.*

Proof. Assume $n > 0$. Using Lemma 1 we associate with a sphere with n handles an integer $k(n)$. Let $k = \max\{3, k(n)\}$. Suppose $p_1, p_2, \cdots, p_{2k}$ are points in $\mathrm{Bd}\, C$ and U is an open set containing $\mathrm{Bd}\, C$. It suffices to show that one of these points has a neighborhood V such that each loop in $V - \mathrm{Bd}\, C$ is nullhomotopic in $U - \mathrm{Bd}\, C$.

Step 1. *Preliminary constructions.* There exists a collection of mutually exclusive disks $D_1, \cdots, D_{2k}$ on $\mathrm{Bd}\, C$ with $p_i \in \mathrm{Int}\, D_i$ $(i = 1, \cdots, 2k)$. Furthermore, $\mathrm{Bd}\, C$ contains another collection of mutually exclusive disks $E_1, \cdots, E_k$ such that for $i = 1, \cdots, k$

$$D_{2i-1} \cup D_{2i} \subset \mathrm{Int}\, E_i\ .$$

We consider C to be embedded in S^3 so that the closure of $S^3 - C$ is a 3-cell [8, 10]. We select a point b of $\mathrm{Int}\, C$ and construct arcs $B_1, \cdots, B_{2k}$ such that (1) distinct arcs B_i and B_j intersect only at the point b, (2) the endpoints of B_i are b and p_i, and (3) B_i is locally tame mod p_i (i $= 1, \cdots, 2k$).

By Theorem 1 of [3] there exist pairwise disjoint annuli

$$D_1^*, D_2^*, \cdots, D_{2k}^*, E_1^*, E_2^*, \cdots, E_k^*$$

in S^3 such that

(4) $\mathrm{Bd}\, D_i^* \supset \mathrm{Bd}\, D_i$ and $\mathrm{Bd}\, E_j^* \supset \mathrm{Bd}\, E_j$,

(5) $D_i^* \cap \mathrm{Bd}\, C \subset D_i$,

(5′) $E_j^* \cap \mathrm{Bd}\, C \subset E_j - (D_{2j-1} \cup D_{2j})$,

(6) $(\cup(\mathrm{Bd}\, D_i^* - \mathrm{Bd}\, D_i)) \cup (\cup(\mathrm{Bd}\, E_j^* - \mathrm{Bd}\, E_j)) \subset \mathrm{Int}\, C$,

(7) $D_i^*(E_j^*)$ is locally polyhedral mod $\mathrm{Bd}\, D_i$ ($\mathrm{Bd}\, E_j$), and

(8) $((\cup D_i^*) \cup (\cup E_j^*)) \cap (\cup B_i) = \varnothing$.

If a surface approximating $\mathrm{Bd}\, C$ is to intersect the D_i^*'s and E_j^*'s properly, we must force it to lie very close to $\mathrm{Bd}\, C$. To do this, first we thicken certain subsets of $\mathrm{Bd}\, C$, thereby obtaining mutually exclusive open sets $W_0, W_1, \cdots, W_{3k}$ such that

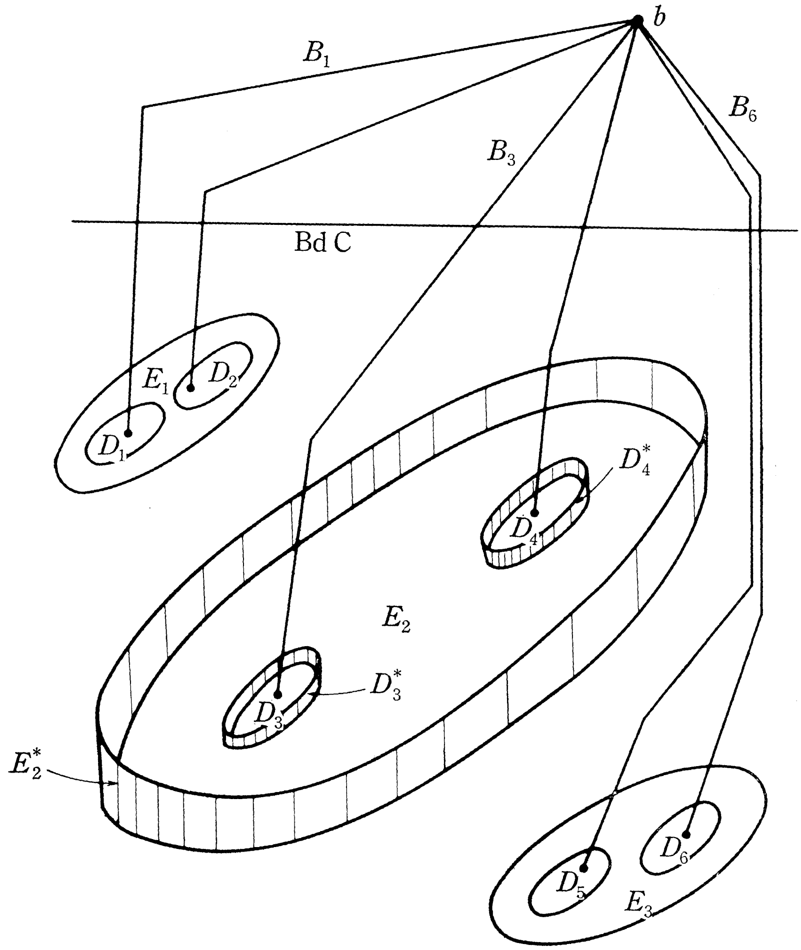

FIGURE 1

(9) $W_i \cap C \subset U - ((\cup \mathrm{Bd}\, D_i^*) \cup (\cup \mathrm{Bd}\, E_j^*))$,
(10) $W_0 \supset \mathrm{Bd}\, C - ((\cup D_i) \cup (\cup E_j))$,
(11) $W_i \supset \mathrm{Int}\, D_i$ $(i = 1, \cdots, 2k)$,
(12) $W_{2k+i} \supset \mathrm{Int}\, E_i - (D_{2i-1} \cup D_{2i})$ $(i = 1, \cdots, k)$,
(13) $(\cup W_j) \cap B_i = W_i \cap B_i$ $(i = 1, \cdots, 2k)$.

In addition, we require that $\mathrm{Bd}\, D_i \cap \mathrm{Cl}\, W_s \neq \emptyset$ only if $s = 2k + i$ or $s = i$ and $\mathrm{Bd}\, E_j \cap \mathrm{Cl}\, W_s \neq \emptyset$ only if $s = 0$ or $s = 2k + j$. Then we construct a neighborhood Y of $\mathrm{Bd}\, C - \cup W_i$ such that $Y \cap C \subset U$ and any arc in $\mathrm{Int}\, C \cap (Y \cup (\cup W_i))$ from a point of W_i to a point of W_j intersects all the annuli in between. For example, if A is an arc from W_0 to W_1, then A intersects both E_1^* and D_1^*.

By hypothesis $\mathrm{Int}\, C$ contains a cube with n holes M such that $C - (Y \cup (\cup W_i)) \subset \mathrm{Int}\, M$. Without loss of generality, we assume that $\mathrm{Bd}\, M$ is polyhedral and in general position with respect to

$$(\cup \mathrm{Int}\, E_j^*) \cup (\cup \mathrm{Int}\, D_i^*) .$$

Step 2. *A special disk in* Bd M. Let G denote the collection of those components of $\mathrm{Bd}\, M \cap (\cup E_j^*) \cup (\cup D_i^*))$ which are essential simple closed curves in any annulus E_j^* or D_i^*. Each annulus $E_j^*(D_i^*)$ contains a curve in the collection G, because $\mathrm{Bd}\, M$ separates the components of $\mathrm{Bd}\, E_j^*(\mathrm{Bd}\, D_i^*)$.

In the next paragraphs we show that at least one of the curves in G bounds a disk in $\mathrm{Bd}\, M$. Suppose the contrary. From Lemma 1 we find that $\mathrm{Bd}\, M$ contains an annulus A such that $\mathrm{Bd}\, A = J_r \cup J_s$, where J_r and J_s are essential curves on E_r^* and E_s^*, respectively, and $r \neq s$. This reduces to the case in which each component of $\mathrm{Int}\, A \cap (\cup E_j^*)$ bounds a disk in $\cup E_j^*$. Assume $r \neq 1 \neq s$.

Case A. *No component of* $A \cap (\cup E_j^*)$ *separates the components of* Bd A. Let L be a simple closed curve in $S^3 - (E_1^* \cup E_r^*)$ such that $L \cap C = B_2 \cup B_{2r}$. It follows from the constructions of Step 1 that each point of $L \cap A$ is separated (in A) from J_s by a component of $\mathrm{Int}\, A \cap (E_1^* \cup E_r^*)$; thus, by trading certain disks in $\mathrm{Int}\, A$ for disks in $E_1^* \cup E_r^*$, we see that J_r and J_s are homotopic in $S^3 - L$. But this is impossible, since J_r links L and J_s does not.

Case B. *Some component of* $A \cap (\cup E_j^*)$ *separates the components of* Bd A. By considering all components of $A \cap ((\cup E_j^*) \cup (\cup D_i^*))$, we find that A contains an annulus A' such that no curve in

$$\mathrm{Int}\, A' \cap ((\cup E_j^*) \cup (D_i^*))$$

is essential in A' and $J_r \subset \mathrm{Bd}\, A'$. Let J' denote the other component of $\mathrm{Bd}\, A'$, and without loss of generality assume that $J' \cap D_{2r}^* = \emptyset$.

Let L' be a simple closed curve in $S^3 - ((\cup E_j^*) \cup (\cup D_i^*))$ such that $L' \cap C = B_2 \cup B_{2r}$. Each point of $L' \cap A'$ is separated in A' from either J_r or J' by $\text{Int}\, A'((\cup E_j^*) \cup (\cup D_i^*))$, and each curve of this intersection bounds disks in both A' and $(\cup E_j^*) \cup (\cup D_i^*)$. Hence, by the usual disk trading, we see that J_r is homotopic to J' in $S^3 - L'$. Again this leads to a contradiction, for J_r links L'; on the other hand, J' either is contained in D_{2r-1}^* or is an inessential curve in some E_j^*, which implies that J' does not link L'.

Neither of the two cases can occur. Consequently, some simple closed curve J in the collection G bounds a disk in $\text{Bd}\, M$.

Step 3. *A neighborhood V of one of the points p_i.* Corresponding to one of the points, say p_1, there exists a disk $D \subset \text{Bd}\, M$ such that $\text{Bd}\, D$ is an essential curve in D_1^*, but each component of Int $D \cap (\cup D_i^*)$ bounds a disk in $\cup D_i^*$. Repeating this process, it follows that for one of the p_i's, say p_1 again, and for each open set U' containing $\text{Bd}\, C$, there exists a polyhedral disk E in $U' \cap \text{Int}\, C$ such that $\text{Bd}\, E$ is an essential simple closed curve on D_1^* but each component of $(\text{Int}\, E \cap (\cup D_i^*))$ bounds a disk in $\cup D_i^*$.

To find the desired open set in C, let V' be a spherical neighborhood of p_1 such that $V' \cap C \subset W_1$, and define $V = V' \cap C$. For any loop L in $V - \text{Bd}\, C$, another linking argument shows that L is separated from $\text{Bd}\, C$ (in V) by some disk $E \subset U$ as described above. Since L is contractible in V', it follows from [5, Lemma 1] that L is contractible in $U - \text{Bd}\, C$. This completes the proof.

THEOREM 3. *Suppose C is a crumpled cube such that* $\text{LG}(\text{Int}\, C) < \infty$ *and C contains at most one nonpiercing point. Then* Int C *is an open 3-cell.*

Proof. Assume C is embedded in S^3 so that the closure of $S^3 - C$ is a 3-cell K [8, 10]. Equivalently, we show that K is a cellular subset of S^3.

Let Q denote the finite set of points of $\text{Bd}\, C$ given by Theorem 2, p the nonpiercing point of C (the argument when C has no nonpiercing point is essentially the same), and U an open set containing K. There exists an open set V containing K such that loops in $V - K$ are null-homotopic in $U - (\text{Int}\, K \cup p)$. Let f be a map of a disk Δ into $U - (\text{Int}\, K \cup p)$ such that $f(\text{Bd}\, \Delta) \subset V - K$. It follows from [12, Th. 2] and techniques of [2, Th. 4.2] that f can be adjusted slightly at points of $\text{Int}\, \Delta$ so that $f(\Delta) \cap \text{Bd}\, C$ is 0-dimensional and $f(\Delta) \cap Q = \varnothing$. Finally, there exists a finite number of mutually exclusive simple closed curves $S_1, \cdots S_k$ in Δ whose union separates $\text{Bd}\, \Delta$ from $f^{-1}(f(\Delta)) \cap \text{Bd}\, C)$ and such that $f | S_i$ is null homotopic in

$U - K$ $(i = 1, \cdots, k)$. This implies that $f|\text{Bd}\,\Delta$ extends to a map of Δ into $U - K$. According to McMillan's Cellularity Criterion [11, Th. 1′], K is a cellular subset of S^3.

3. Topological collapsing. The following result generalizes Theorem 1 of [13]. The argument below necessarily differs from McMillan's, since we have no mapping criterion to determine the finite limiting genus condition.

THEOREM 4. *Suppose K is a finite connected simplicial complex, L a subcomplex of K such that K collapses to L, and h a homeomorphism of K into S^3 such that* $\text{LG}(S^3 - h(K)) = n$. *Then*

$$\text{LG}(S^3 - h(L)) \leqq n\ .$$

Proof. It is sufficient to show that the result holds if L is obtained from K by a single elementary collapse. Suppose that σ is a principal simplex of K, τ is a proper face of σ such that τ is a proper face of no other simplex in K, and

$$L = K - \text{Int}\,\sigma - \text{Int}\,\tau\ .$$

We consider the case when σ is a 3-simplex, because the applications of Theorem 4 in this paper can be viewed as involving collapses of this type only; for the remaining cases a similar argument applies.

Let U be an open subset of S^3 containing $h(L)$. There exists a neighborhood U^* of $h(L)$ in U such that some component Z of $h(\sigma) - U^*$ contains $h(\sigma) - U$. Using [4, Th. 4] we find a tame disk D in $U^* - h(L)$ such that $\text{Bd}\,D \cap h(K) = \varnothing$ and exactly one of the components of $D \cap h(\sigma)$ separates Z from $h(L \cap \sigma)$ in $h(\sigma)$.

There exists a neighborhood W of $h(K)$ such that $W \cap \text{Bd}\,D = \varnothing$ and W can be deformed to $h(K)$ in S^3-$\text{Bd}\,D$ by a homotopy keeping $h(K)$ pointwise fixed. For each point x in $U \cap h(K)$ define an open set N_x as

$$N_x = \{y \in S^3 \mid \rho(x, y) < \rho(x, \text{Bd}\,U \cup \text{Bd}\,W)\}$$

and for each point x in $h(\sigma) - U$ define N_x as

$$N_x = \{y \in S^3 \mid \rho(x, y) < \rho(x, D \cup \text{Bd}\,W)\}\ .$$

Then let $V = \bigcup_{x \in h(K)} N_x$.

Claim. $D \cap V$ separates Z from $h(L)$ in V, and U contains the component Y of $V - D$ that contains $h(L)$.

Suppose there exists an arc α in $V - D$ from a point of Z to a point of $h(L)$. Then α is homotopic in $S^3 - \operatorname{Bd} D$ (with endpoints fixed) to a path α' in $h(K)$, and α' is homotopic in $h(K)$ (with endpoints fixed) to a path α^* such that $\alpha^* \cap D$ consists of a finite set of points at which α^* pierces D. But then the number of such points must be even, contradicting the separation properties of D in $h(K)$.

To establish the other part of the claim, suppose there exists a point y in $Y - U$. Then $y \in N_x$ for some x in $h(\sigma) - U$. Let A be the straight line segment from y to x in N_x, and let B denote an arc from y to $h(L)$ in Y. Since $A \cup B$ does not intersect D, deforming $A \cup B$ to a path in $h(K)$ leads to a contradiction as before. This completes the proof of the claim.

By hypothesis $S^3 - h(K)$ contains a polyhedral cube with n holes H such that $\operatorname{Int} H \supset S^3 - V$. We adjust H slightly so that $\operatorname{Bd} H \cap D$ consists of a finite number of simple closed curves. Note that $D \cup (\operatorname{Bd} H \cap U)$ separates $h(L)$ from $h(\sigma) - U$ (in S^3). Thus, the unicoherence of $S^3 - D$ implies that some component F of $\operatorname{Bd} H - D$, where $F \subset U$, separates $h(L)$ from $h(\sigma) - U$ in $S^3 - D$.

We observe that $\operatorname{Cl} F$ is a disk with k ($k \leqq n$) handles and (possibly) some holes. By attaching disks to $\operatorname{Bd} F$ near D, we see that F is contained in a sphere with k handles S_k in $\operatorname{Cl}(S^3 - h(L))$ and that S_k bounds a cube with k holes M satisfying

$$S^3 - U \subset M \subset S^3 - h(L) .$$

This implies that $\operatorname{LG}(S^3 - h(L)) \leqq n$.

4. The number of nonpiercing points.

THEOREM 5. *If C is a crumpled cube such that* $\operatorname{LG}(\operatorname{Int} C) = n$ $(1 \leqq n < \infty)$, *then C has at most n nonpiercing points.*

Proof. Suppose to the contrary that C contains at least $n+1$ nonpiercing points $p_1, \cdots, p_{n+1}$. As before we assume C is embedded in S^3 so that the closure of S^3 of $S^3 - C$ is a 3-cell H [8, 10]. Let h denote a homeomorphism of a 3-simplex Δ^3 onto H.

Some triangulation K of Δ^3 collapses to a subcomplex L such that $h(L)$ is a 3-cell locally tame except at $p_1, \cdots, p_{k+1}$; thus, each point p_i is a nonpiercing point of $\operatorname{Cl}(S^3 - h(L))$. Theorem 4 gives that $\operatorname{LG}(S^3 - h(L)) \leqq n$. This leads to a contradiction, however, for either [6, Th. 2] or [14, Th. 1] implies that $\operatorname{Cl}(S^3 - h(L))$ has at most n nonpiercing points.

COROLLARY. *If C is a crumpled cube such that* $\operatorname{LG}(\operatorname{Int} C) \leqq 1$, *then* $\operatorname{Int} C$ *is an open* 3-*cell.*

The techniques used to prove Theorem 5 can be reapplied to obtain the following result.

THEOREM 6. *If H is a cube with k handles in S^3 and*

$$\mathrm{LG}(S^3 - H) = n \, (1 \leqq n < \infty) ,$$

then Bd H *is pierced by a tame arc at all but (at most) $n - k$ of its points.*

To describe the number of nonpiercing points precisely requires some additional definitions. Let A be an arc in S^3 locally tame modulo an endpoint p. The *local enveloping genus of A at p*, denoted LEG (A, p), is the smallest nonnegative integer r (if there is no such integer r, $\mathrm{LEG}(A, p) = \infty$) such that there exist arbitrarily small neighborhoods of p, each of which is bounded by a surface of genus r (a sphere with r handles) that intersects A at exactly one point. Chapter 4 of [14] gives illustrations of arcs A_n, each locally tame mod an endpoint p_n, such that $\mathrm{LEG}(A_n, p_n) = n$ $(n = 1, 2, \cdots, \infty)$.

Let $B = \{(x, y, z) \in E^3 \,|\, x^2 + y^2 + z^2 \leqq 1\}$. Let f be a homeomorphism of B onto a 3-cell C in S^3, and p a point of Bd C. The *local enveloping genus of C at p*, denoted $\mathrm{LEG}(C, p)$, is defined by

$$\mathrm{LEG}(C, p) \doteq \mathrm{LEG}(f(\alpha), p) ,$$

where α is the line segment in B from the origin to $f^{-1}(p)$.

THEOREM 7. *If C is a 3-cell in S^3 such that* $\mathrm{LG}(S^3 - C) = n$ $(2 \leqq n < \infty)$ *and $p_1, \cdots, p_k$ are the nonpiercing points of* $S^3 - \mathrm{Int}\, C$, *then*

$$n = \sum_{i=1}^{k} \mathrm{LEG}(C, p_i) .$$

Proof. As in the proof of Theorem 5, let h be a homeomorphism of a 3-simplex Δ^3 onto C. Some triangulation of Δ^3 collapses to a subcomplex L such that $h(L)$ is a 3-cell locally tame modulo $\cup p_i$. It follows from the definition of local enveloping genus that the subcomplex L can be chosen to satisfy

$$\mathrm{LEG}(C, p_i) = \mathrm{LEG}(h(L), p_i) \qquad (i = 1, \cdots, k) .$$

Since $\mathrm{LG}(S^3 - h(L)) \leqq n$, Theorem 6 of [14] implies

$$n \geqq \Sigma\, \mathrm{LEG}(h(L), p_i) = \Sigma\, \mathrm{LEG}(C, p_i) .$$

Let U be an open set containing C. To establish the inequality in the other direction, we shall find pairwise disjoint disks with handles

$G_1, \cdots, G_k$ in $U - \cup p_i$ subject to the following conditions: the number of handles on G_i is bounded by LEG(C, p_i), $\mathrm{Bd}\, G_i$ bounds an annulus A_i in G_i such that $G'_i = \mathrm{Cl}\,(G_i - A_i)$ is contained in $U - C$, $\mathrm{Int}\, A_i \cap \mathrm{Bd}\, C$ is contained both in a null sequence of pairwise disjoint disks in $\mathrm{Bd}\, C - \cup p_i$ and in a null sequence of such disks in $\mathrm{Int}\, A_i$, and $\cup \mathrm{Bd}\, G_i$ bounds a disk with $(k-1)$ holes in $\mathrm{Bd}\, C - \cup p_i$. Furthermore, G_i can be obtained arbitrarily close to p_i. Thus, in the next two paragraphs we describe how to find one such surface G_1 near p_1.

In $\mathrm{Bd}\, C$ there exists a Sierpinski curve X locally tame mod p_1 and containing p_1 in its inaccessible part. By removing a null sequence of nice 3-cells from C we obtain a 3-cell C^* such that $C^* \cap \mathrm{Bd}\, C = X$ and C^* is locally tame mod p_1. It follows from the definition of local enveloping genus that arbitrarily close to p_1 is a surface H such that $H \cap C^*$ is a disk D, with $D \cap \mathrm{Bd}\, C^* = \mathrm{Bd}\, D$, and p_1 lies interior to the small disk on $\mathrm{Bd}\, C^*$ bounded by $\mathrm{Bd}\, D$. Adjust H near $\mathrm{Bd}\, C^*$ so that $\mathrm{Bd}\, D$ lies in the inaccessible part of X. Without moving any point of D adjust H further so that the nondegenerate components of $(H - D) \cap \mathrm{Bd}\, C$ comprise a null sequence of simple closed curves and that $(H - D) \cap C^* = \emptyset$ [4, Th. 4]. Hence,

$$(H - D) \cap X = \emptyset \,.$$

Now consider the component K of $H - C$ whose closure contains $\mathrm{Bd}\, D$. Associate with each simple closed curve S_j of $(\mathrm{Bd}\, K - \mathrm{Bd}\, D)$ a disk F_j in $C - C^*$ such that

(1) $F_j \cap \mathrm{Bd}\, C = \mathrm{Bd}\, F_j = S_j$,

(2) $F_j \cap F_k = \emptyset$ if $S_j \cap S_k = \emptyset$,

(3) $\lim_{j \to \infty} \mathrm{diam}\, F_j = 0$.

Define $G_1 = (\cup F_j) \cup \mathrm{Cl}\, K$. Then G_1 is a disk with handles, and the number of handles is bounded by LEG(C, p_1). Note that $\mathrm{Bd}\, G_1 = \mathrm{Bd}\, D$. Since components of $(G_1 - \mathrm{Bd}\, G_1) \cup C$ are either arcs or points, we can readily obtain an annulus A_1 in G_1 such that $\mathrm{Bd}\, A_1$ contains $\mathrm{Bd}\, G_1$ and $\mathrm{Int}\, A_1$ contains $(G_1 - \mathrm{Bd}\, G_1) \cap C$, and now the remaining requirements on G_1 must be satisfied.

Applying Theorem 2 and techniques from the proof of Theorem 3, we find a map f of a disk with $(k-1)$ holes E into $U - C$ such that

$$f(E) \cap G'_i = f(\mathrm{Bd}\, E) \cap G'_i = \mathrm{Bd}\, G'_i \qquad (i = 1, \cdots, k)$$

and f has no singularities near $\mathrm{Bd}\, E$. According to [9, Lemma 1] there exists a homeomorphism f' of E into $U - C$ such that

$$f'(E) \cap G'_i = f'(\mathrm{Bd}\, E) \cap G'_i = \mathrm{Bd}\, G'_i \quad (i = 1, \cdots, k) \,.$$

Thus, if S denotes $f'(E) \cup (\cup G'_i)$, S is a sphere with handles, and the number of handles is bounded by $\Sigma\,\mathrm{LEG}(C, p_i)$. Moreover, S can be obtained so as to separate $S^3 - U$ from C. Finally, since U is an arbitrary open set, we have that

$$n \leqq \sum \mathrm{LEG}(C, p_i) .$$

5. Semi-cellular subsets.

THEOREM 8. *Suppose C is a crumpled cube such that*

$$2 \leqq \mathrm{LG}(\mathrm{Int}\, C) < \infty ,$$

and X is a nonseparating subcontinuum of Bd C *containing only piercing points of C. Then X is semi-cellular in C.*

Proof. Let $p_1, \cdots, p_k$ denote the nonpiercing points of C, and D a disk in $\mathrm{Bd}\, C - \cup p_i$ whose interior contains X. If C is embedded in S^3 so that $\mathrm{Cl}\,(S^3 - C)$ is a 3-cell K, then K collapses to a 3-cell K' which is locally tame mod $(D \cup p_1)$, with p_1 a nonpiercing point of $S^3 - \mathrm{Int}\, K' = C'$. According to Theorem 4, $\mathrm{LG}(\mathrm{Int}\, C') < \infty$. Since each point of D is a piercing point of C', it follows from Theorem 3 that $\mathrm{Int}\, C'$ is an open 3-cell. Then X is semi-cellular in C' [7, Lemma 2.7]; clearly X must also be semi-cellular in C.

Theorem 8 can be applied to characterize those sewings of two crumpled cubes which yield S^3, when one of the crumpled cubes has finite limiting genus. With minor changes, such as in the references to the number of nonpiercing points, we can use the proof of [7, Th. 5.7] to prove Theorem 9.

THEOREM 9. *Suppose C_1 and C_2 are crumpled cubes, h is a homeomorphism of* Bd C_1 *to* Bd C_2, *and* $\mathrm{LG}(\mathrm{Int}\, C_2) < \infty$. *Then $C_1 \bigcup_h C_2 = S^3$ if and only if each nonpiercing point of C_1 is identified by h with a piercing point of C_2.*

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UNIVERSITY OF TENNESSEE