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PROJECTING ONTO CYCLES IN SMOOTH, REFLEXIVE BANACH SPACES

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This paper deals with operator algebras generated by certain classes of norm 1 projections on smooth, reflexive Banach spaces. For a strictly increasing continuous function $\mathscr{F}$ on the nonnegative reals, the set of "$\mathscr{F}$-projections" gives rise to operator algebras equal to their second commutants. The principal result is that the closed subspace generated by the set of elements Ex, where x is fixed and E runs through a Boolean algebra of $\mathscr{F}$-projections, is the range of a norm 1 projection that commutes with each projection in the Boolean algebra. Sufficient conditions using Clarkson type norm inequalities are given for the commutativity of the set of all $\mathscr{F}$-projections. Examples in Orlicz spaces are given.

1. Projections in smooth spaces. A *normer* of a nonzero element x in a Banach space X is a functional x^* in the dual X^* such that $||x^*|| = 1$ and $||x|| = x^*(x)$. A normer for x always exists; we say that X is *smooth* if every nonzero x has but one normer, denoted $N(x)$. We make the definition $N(0) = 0$.

Proof of the following three lemmas is left to the reader; see, for instance, [5; p. 447].

LEMMA 1. *In a smooth space X, the norming map $N: X \to S^* \cup \{0\}$ has the following properties, where S^* is the unit sphere of X^*.*

(1) *$N(x)$ is the only element of S^* such that $N(x)(x) = ||x||$ if $x \neq 0$.*

(2) *$N(\lambda x) = (|\lambda|/\lambda)N(x)$ for all scalars $\lambda \neq 0$; in particular, $N(\lambda x) = N(x)$ for $\lambda > 0$.*

(3) *In the real case, $N(x)(y) = \lim(\lambda \to 0)(||x + \lambda y|| - ||x||)/\lambda$ for $x, y \in X$ and $x \neq 0$.*

LEMMA 2. *If X is a smooth complex Banach space,* Re X *is also smooth; indeed, for each $x \neq 0$,* Re $N(x)$ *is the normer of x in* (Re X)*.

A vector x is said to be *James-orthogonal* to y if $||x + \lambda y|| \geqq ||x||$ for all real numbers λ.

LEMMA 3. *If X is a smooth space, then $N(x)(y) = 0$ if and only if x is James-orthogonal to y in the real case and James-orthogonal to both y and iy in the complex case. If Y is a subspace, then $N(x)(y) = 0 (y \in Y)$ if and only if $||x + y|| \geqq ||x|| (y \in Y)$.*

LEMMA 4. *If E is a norm one projection in a normed linear space X, then $\|a + b\| \geqq \|a\|$ for every $a \in EX$ and $b \in (I - E)X$.*

Proof. $\|a\| = \|E(a + b)\| \leqq \|a + b\|$.

LEMMA 5. *If E is a norm one projection on a smooth space X, $N(Ex)(Ey) = N(Ex)(y)(x, y \in X)$.*

Proof. This is an immediate consequence of Lemmas 3 and 4.

THEOREM 6. *A subspace of a smooth space X can be the range of at most one norm 1 projection.*

Proof. Suppose E and F are norm 1 projections on X with $EX = FX$. Then $EF = F$ and $FE = E$ so that $E - F = E(I - F) = F(E - I)$. If $E \neq F$, there is an x such that

$$\begin{aligned} 0 \neq \|Ex - Fx\| &= N(Ex - Fx)(Ex - Fx) \\ &= N(E(I - F)x)(Ex) - N(F(E - I)x)(Fx) \\ &= N(E(I - F)x)(x) - N(F(E - I)x)(x) = 0 , \end{aligned}$$

a contradiction.

We wish to thank the referee for sharpening the following two lemmas into their present form and for suggesting lines of proof.

THEOREM 7. *A subspace of a rotund space can be the null manifold of at most one norm 1 projection.*

Proof. Suppose E and F are distinct norm 1 projections on a rotund space X, with the same null manifold N. Then there is an element x in the range of E that is not in the range of F. Then $x = y + w$ where y is the range of F, w is in N, and x and y are not linearly dependent.

$$\begin{aligned} \|x\| &= \|E(x - 1/2w)\| \leqq \|x - 1/2w\| = \|1/2(x + y)\| \\ \|y\| &= \|F(y + 1/2w)\| \leqq \|y + 1/2w\| = \|1/2(x + y)\| \end{aligned}$$

so that $1/2(\|x\| + \|y\|) \leqq \|1/2(x + y)\| \leqq 1/2(\|x\| + \|y\|)$, $\|x + y\| = \|x\| + \|y\|$, and X is not rotund.

THEOREM 8. *For any norm 1 projection E on a smooth space X, $N(EX \cap S) \subseteqq E^*X^* \cap N(S)$, with equality if X is smooth and rotund. If X is reflexive, then $N(S) = S^*$, but in any case $N(S)$ is dense in S^*.*

Proof. If $x^* \in N(EX \cap S)$, then there is a norm 1 vector x such that $x^* = N(x)$ and $Ex = x$. Then $E^*N(x)(y) = N(Ex)(Ey) = N(Ex)(y) = x^*(y)$ by Lemma 5 for all y in X; hence, $x^* \in E^*X^* \cap N(S)$.

If X is rotund and $x^* \in E^*X^* \cap N(S)$, then $x^* = N(x)$ where $||x|| = 1$ and $E^*(N(x)) = N(x)$. Then

$$\begin{aligned} &||x + Ex|| \leqq ||x|| + ||Ex|| \leqq ||x|| + ||x|| \\ &= N(x)(x) + N(x)(x) = N(x)(x) + (E^*N(x))(x) = N(x)(x + Ex) \leqq ||x + Ex|| \,. \end{aligned}$$

Then $||x|| + ||Ex|| = ||x + Ex||$ and $x = Ex$ by rotundity and the fact that E is a projection.

The last statement follows from results of James [7] and Bishop-Phelps [2].

2. $\mathscr{F}$-projections.

Throughout this section, $\mathscr{F}$ denotes a fixed, but arbitrary, strictly increasing continuous function from the set of nonnegative real numbers into itself.

DEFINITION. An *$\mathscr{F}$-projection* on a Banach space X is a projection E on X for which $\mathscr{F}(||x||) = \mathscr{F}(||Ex||) + \mathscr{F}(||(I - E)x||)$ for all x in X.

LEMMA 9. (1) *An $\mathscr{F}$-projection has norm* 1 *or* 0; (2) *If E is an $\mathscr{F}$-projection, $\mathscr{F}(||a + b||) = \mathscr{F}(||a||) + \mathscr{F}(||b||)$ and $||a + b|| = ||a - b||$ for all a in $E[X]$,* b in $(I - E)[X]$; (3) *the product of two commuting $\mathscr{F}$-projections is an $\mathscr{F}$-projection.*

Proof. (1) If E is an $\mathscr{F}$-projection,

$$\mathscr{F}(||EX||) \leqq \mathscr{F}(||Ex||) + \mathscr{F}(||(I - E)x||) = \mathscr{F}(||x||) \,.$$

Since $\mathscr{F}$ is strictly increasing, $||Ex|| \leqq ||x||$.

$$\begin{aligned} &\mathscr{F}(||a + b||) = \mathscr{F}(||Ea + (I - E)b||) \\ (2) \quad &= \mathscr{F}(||E(Ea + (I - E)b||) + \mathscr{F}(||(I - E)(Ea + (I - E)b||) \\ &= \mathscr{F}(||Ea||) + \mathscr{F}(||(I - E)b||) \,, \end{aligned}$$

and

$$\begin{aligned} &||a + b|| = \mathscr{F}^{-1}(\mathscr{F}(||a + b||) = \mathscr{F}^{-1}(\mathscr{F}(||a||) + \mathscr{F}(||b||)) \\ &= \mathscr{F}^{-1}(\mathscr{F}(||a||) + \mathscr{F}(||-b||)) = \mathscr{F}^{-1}(\mathscr{F}(||a - b||)) = ||a - b|| \,. \end{aligned}$$

(3) If E and F are commuting $\mathscr{F}$-projections,

$$\begin{aligned} &\mathscr{F}(||x||) = \mathscr{F}(||Fx||) + \mathscr{F}(||(I - F)x||) \\ &= \mathscr{F}(||EFx||) + \mathscr{F}(||(I - E)Fx||) + \mathscr{F}(||(I - F)x||) \end{aligned}$$

$$= \mathscr{F}(||EFx||) + \mathscr{F}(||F(I - E)x + (I - F)x||)$$
$$= \mathscr{F}(||EFx||) + \mathscr{F}(||(I - EF)x||)$$

for all x in X.

REMARK. If E is an $\mathscr{F}$-projection, then $||a + b||$, where a is any norm 1 vector in EX and b is any norm 1 vector in $(I - E)X$, is constant at $\mathscr{F}^{-1}(2\mathscr{F}(1))$. For

$$||a + b|| = \mathscr{F}^{-1}\mathscr{F}(||a + b||) = \mathscr{F}^{-1}(\mathscr{F}(||a||) + \mathscr{F}(||b||) .$$

THEOREM 10. *A maximal family $\mathscr{P}$ of commuting $\mathscr{F}$-projections is a complete-Boolean algebra of norm* 1 *projections.*

Proof. Clearly 0 and I are in $\mathscr{P}$ and if E is in $\mathscr{P}$, so is $I - E$ by the symmetry of the definition of an $\mathscr{F}$-projection. If E and F are in $\mathscr{P}$, EF is an $\mathscr{F}$-projection by Lemma 9, and it commutes with $\mathscr{P}$. Therefore, EF is in $\mathscr{P}$. Thus $\mathscr{P}$ is a Boolean algebra of projections on X as defined by Bade [1]. Now suppose E_α is an increasing net of projections in $\mathscr{P}$. For each x in X and for $\alpha \leqq \beta$, $E_\alpha x = E_\alpha E_\beta x$. So $||E_\alpha x|| \leqq ||x||$; thus, $\mathscr{F}(||E_\alpha x||)$ is an increasing net of real numbers bounded above by $\mathscr{F}(||x||)$; hence, covergent. This implies $E_\alpha x$ is Cauchy, as follows. Given $\varepsilon \geqq 0$, choose θ such that

$$\mathscr{F}(||E_\alpha x||) \geqq \lim_\gamma \mathscr{F}(||E_\gamma x||) - \mathscr{F}(\varepsilon/2)$$

for all $\alpha \geqq \theta$. If $\beta \geqq \theta$,

$$\mathscr{F}(||E_\beta x - E_\theta x||) + \mathscr{F}(||E_\theta x||)$$
$$= \mathscr{F}(||E_\beta x - E_\beta E_\theta x||) + \mathscr{F}(||E_\theta E_\beta x||)$$
$$= \mathscr{F}(||(I - E_\theta)E_\beta x||) + \mathscr{F}(||E_\theta E_\beta x||) = \mathscr{F}(||E_\beta x||) .$$

Thus,

$$\mathscr{F}(||E_\beta x - E_\theta x||) = \mathscr{F}(||E_\beta x||) - \mathscr{F}(||E_\theta x||) .$$

And from this

$$\mathscr{F}(\varepsilon/2) \geqq \lim{}_\alpha \mathscr{F}(||E_\alpha x||) - \mathscr{F}(||E_\theta x||)$$
$$\geqq \mathscr{F}(||E_\beta x||) - \mathscr{F}(||E_\theta x||) = \mathscr{F}(||E_\beta x - E_\theta x||) ;$$

hence, $\varepsilon/2 \geqq ||E_\beta x - E_\theta x||$ because $\mathscr{F}$ is increasing. If $\alpha, \beta \geqq \theta$,

$$||E_\alpha x - E_\beta x|| \leqq ||E_\alpha x - E_\theta x|| + ||E_\beta x - E_\theta x|| \leqq \varepsilon .$$

Define $Ex = \lim_\alpha E_\alpha x$ for every x in X. Then E is surely a projection and, since $\mathscr{F}$ is continuous, E is an $\mathscr{F}$-projection; since E

commutes with $\mathscr{P}$, it is in $\mathscr{P}$. This completes the argument.

By Zorn's lemma, complete Boolean algebras of $\mathscr{F}$-projections always exist, although they may be trivial. Nontrivial examples are given later.

THEOREM 11. *Suppose that all vectors v and w in X satisfy the (Clarkson) inequality*

$$1/2\mathscr{F}(\|v+w\|) + 1/2\mathscr{F}(\|v-w\|) \leqq \mathscr{F}(\|v\|) + \mathscr{F}(\|w\|)$$

and suppose $\mathscr{F}(2) \neq 4$, $\mathscr{F}(1) = 1$. Then any two $\mathscr{F}$-projections commute (and so the set of all $\mathscr{F}$-projections form a complete Boolean algebra of projections). The same result holds for the reverse inequality.

Proof. Let E and F be two $\mathscr{F}$-projections and $x \in X$. Then decomposing Ex into F and then E components, applying Clarkson's inequality, and simplifying (using Lemma 9) we obtain

$$\begin{aligned}\mathscr{F}(\|Ex\|) &= \mathscr{F}(\|EFEx\|) + \mathscr{F}(\|E(I-F)Ex\|) \\ &\quad + \mathscr{F}(\|(I-E)FEx\|) + \mathscr{F}(\|(I-E)(I-F)Ex\|) \\ &\geqq 1/2\mathscr{F}(\|EFEx + E(I-F)Ex)\|) + 1/2\mathscr{F}(\|EFEx - E(I-F)Ex\|) \\ &\quad + 1/2\mathscr{F}(\|(I-E)FEx + (I-E)(I-F)Ex\|) \\ &\quad + 1/2\mathscr{F}(\|(I-E)FEx - (I-E)(I-F)Ex\|) \\ &= 1/2\mathscr{F}(\|Ex\|) + 1/2\mathscr{F}(\|EFEx - E(I-F)Ex \\ &\quad + (I-E)FEx - (I-E)(I-F)Ex\|) \\ &= 1/2\mathscr{F}(\|Ex\|) + 1/2\mathscr{F}(\|FEx - (I-F)Ex\|) \\ &= 1/2\mathscr{F}(\|Ex\|) + 1/2\mathscr{F}(\|FEx + (I-F)Ex\|) \\ &= \mathscr{F}(\|Ex\|) .\end{aligned}$$

This implies equality in Clarkson's inequality for the vectors $(I-E)FEx$ and $(I-E)(I-F)Fx$:

$$\begin{aligned}&\mathscr{F}(\|(I-E)FEx\|) + \mathscr{F}(\|(I-E)(I-F)Ex\|) \\ &= 1/2\mathscr{F}(\|(I-E)FEx + (I-E)(I-F)Ex\|) \\ &\quad + 1/2\mathscr{F}(\|(I-E)FEx - (I-E)(I-F)Ex\|) .\end{aligned}$$

Since the first term on the right is zero, we can define $Z \equiv Z(x) \equiv (I-E)FEx \equiv -(I-E)(I-F)Ex$ and obtain $4\mathscr{F}(\|z\|) = \mathscr{F}(2\|z\|)$. What if $Z(x) \neq 0$? Then $\|Z(x/\|Z(x)\|)\| = 1$, and we have

$$4 = 4\mathscr{F}(\|Z(x/\|Z(x)\|)\|) = \mathscr{F}(2\|Z(x/\|Z(x)\|)\|) = \mathscr{F}(2)$$

which contradicts the hypothesis. Thus $Z = 0$ and so $FEx = EFEx$

for any x and any two $\mathscr{F}$-projections E and F. Replacing E and F by $(I - E)$ and F yields $F(I - E)x = (I - E)F(I - E)x$; whence $EFx = EFEx$. Therefore $FEx = EFx$ and so E and F commute.

REMARK. Consider $\mathscr{F}(\lambda) = \lambda^p$ for a fixed $p, 1 \leqq p < \infty$. An $\mathscr{F}$-projection for such an $\mathscr{F}$ is called an L^p-projection. Cunningham [4] showed that the L^1 projections always commute in any Banach space. The above theorem shows that for $p \neq 2$, the L^p projections in an L^p space commute.

DEFINITION. A net T_α of projections on a Banach space X is said to be *increasing* if $\alpha < \beta$ implies $T_\alpha T_\beta = T_\alpha = T_\beta T_\alpha$.

THEOREM 12. *If T_α is an increasing net of norm 1 projections on a reflexive Banach space X, then T_α converges in the strong opertor topology of X to a norm 1 projection T that commutes with each T_α and whose range is the norm closure of $\bigcup_\alpha T_\alpha[X]$.*

Proof. The essentials of a proof can be found in [8; p. 223].

3. Projecting onto cycle subspaces.

DEFINITION. If $\mathscr{P}$ is a Boolean algebra of projections on X and x is in X, let $S(x; \mathscr{P})$ denote the *cycle generated by x and $\mathscr{P}$*; that is, the closed subspace of X generated by $\{Ex: E \in \mathscr{P}\}$.

THEOREM 13. *Let $\mathscr{P}$ be a Boolean algebra of $\mathscr{F}$-projections on a Banach space X that is smooth and reflexive, and let $x \in X$. Then $S(x; \mathscr{P})$ is the range of a (unique) norm 1 projection that commutes with $\mathscr{P}$.*

Proof. Let π denote the set of all partitions of x by $\mathscr{P}$; that is, finite subsets $\{E_1, \cdots, E_n\}$ of $\mathscr{P}$ such that $E_iE_j = 0$ if $i \neq j$ and $(V_iE_i)(x) = \sum_i E_ix = x$. The set $\{I\}$ is such a partition. Order π by setting $\mathscr{E} r \mathscr{A}$ if, given A in $\mathscr{A}$ there is an E in $\mathscr{E}$ such that $AE = A$. This "is refined by" relation r is reflexive, anti-symmetric, transitive, and it directs the set π. Indeed, if $\{E_1, \cdots, E_n\}$ and $\{A_1, \cdots, A_m\}$ are partitions of x, then one common refinement is the set of E_iA_j such that $E_iA_jx \neq 0$.

For each partition $\mathscr{E}$ of x, define $T(\mathscr{E})(y) \equiv \sum (E \in \mathscr{E})(N(Ex)(y)/\|Ex\|)Ex$ for all y in X. The transformation $T(\mathscr{E})$ is obviously linear; that it is a projection on X is an immediate consequence of the fact that for E and F in $\mathscr{P}$ with $EF = 0$, $N(Ez)(Fy) = N(Ez)(EFy) = 0$. We now show that the norm of $T(\mathscr{E})$ is 1. It is not 0, first of all, because the projection leaves x fixed. Proceeding, let $y \in X$.

$$||[N(Ex)(y)/||Ex||]Ex|| = |N(Ex)(y)| = |N(Ex)(Ey)| \leqq ||Ey|| .$$

From this,

$$\begin{aligned} &\mathscr{F}(||y||) \geqq \mathscr{F}(||V(E \in \mathscr{E})Ey||) = \mathscr{F}(||\sum (E \in \mathscr{E})Ey||) \\ &= \sum (E \in \mathscr{E})\mathscr{F}(||Ey||) \geqq \sum (E \in \mathscr{E})\mathscr{F}(||N(Ex)(y)/||Ex||)Ex||) \\ &= \mathscr{F}(||\sum (E \in \mathscr{E})(N(Ex)(y)/||Ex||)Ex||) = \mathscr{F}(||T(\mathscr{E})y||) . \end{aligned}$$

Consequently $||T(\mathscr{E})y|| \leqq ||y||$.

In order to apply Theorem 12, we must show that $T(\mathscr{A})T(\mathscr{E}) = T(\mathscr{E}) = T(\mathscr{E})T(\mathscr{A})$ under the assumption that $\mathscr{E} r \mathscr{A}$. It is a routine matter to use Lemma 5 to check that $T(\mathscr{A})(Ax) = Ax$ for any A in $\mathscr{A}$, that $T(\mathscr{A})(Ex) = Ex$ for any E in $\mathscr{E}$, and that, therefore, $T(\mathscr{E}) = T(\mathscr{A})T(\mathscr{E})$. Let z be a given element of the null manifold of $T(\mathscr{A})$. Then for each A in $\mathscr{A}$, $(N(Ax)(z)/||Ax||)Ax = AT(\mathscr{A})z = 0$ so that $N(Ax)(Az) = N(Ax)(z) = 0$. Then Ax is James orthogonal to Az:

$$||Ax + Az|| \geqq ||Ax|| .$$

Then

$$\begin{aligned} &\mathscr{F}(||Ex + Ez||) = \mathscr{F}(||(\sum (AE = A)A(x + z)||) \\ &= \sum (AE = A)\mathscr{F}(||Ax + Az||) \geqq \sum (AE = A)\mathscr{F}(||Ax||) \\ &= \mathscr{F}(||\sum (AE = A)Ax||) = \mathscr{F}(||Ex||) , \end{aligned}$$

for every E in $\mathscr{E}$. Therefore, $||Ex + Ez|| \geqq ||Ex||$ and, similarly, $||Ex + iEz|| \geqq ||Ex||$ if X is complex. In any case, $N(Ex)(z) = N(Ex)(Ez) = 0$ for all E in $\mathscr{E}$ and, therefore, z is in the null manifold of $T(\mathscr{E})$. Since the null manifold of $T(\mathscr{E})$ contains that of $T(\mathscr{A})$, we have $T(\mathscr{E})T(\mathscr{A}) = T(\mathscr{E})$.

By Theorem 12, there is a norm 1 projection T commuting with every $T(\mathscr{E})$ that is the limit in the strong operator topology of the net $T(\mathscr{E})$ and whose range is the subspace $\mathrm{cl} \bigcup (\mathscr{E} \in \pi) T(\mathscr{E})[X]$. Let us show that T commutes with the projections in $\mathscr{P}$. Let $E \in \mathscr{P}$. If $Ex \neq 0$, let $\mathscr{E}$ denote the set $\{E\}$ or $\{E, I - E\}$ that is a partition of x. Given $\mathscr{A} \in \pi$ such that $\mathscr{E} r \mathscr{A}$,

$$\begin{aligned} T(\mathscr{A})Ey &= \sum (A \in \mathscr{A})(NAx)(Ey)/||Ax||)Ax \\ &= \sum (AE = A)(N(Ax)Ey)/||Ax||)Ax \\ &= \sum (AE = A)(N(Ax)(y)/||Ax||)EAx \\ &= E(\sum (AE = A)(N(Ax)(y)/||Ax||)Ax) \\ &= E(\sum (A \in \mathscr{A})(N(Ax)(y)/||Ax||)Ax) \\ &= ET(\mathscr{A})y \end{aligned}$$

for all y in X. Consequently, for each y in X,

$$\begin{aligned} TEy &= \lim (\mathscr{E} r \mathscr{A}) T(\mathscr{A}) Ey = \lim (\mathscr{E} r \mathscr{A}) ET(\mathscr{A}) y \\ &= E \lim (\mathscr{E} r \mathscr{A}) T(\mathscr{A}) y = ETy \ . \end{aligned}$$

Therefore, $TE = ET$ provided $Ex \neq 0$. If $Ex = 0$, then $(I - E)x \neq 0$ and $T(I - E) = (I - E)T$ by the same argument. From this, $TE = ET$ when $Ex = 0$.

For all $\mathscr{A}$ in π, $T(\mathscr{A})[X] \subseteqq S(x; \mathscr{P})$; hence, $T[X] \subseteqq S(x; \mathscr{P})$. And given $E \in \mathscr{P}$, if $Ex \neq 0$, then, letting $\mathscr{E}$ be the above partition of x, $S(x; \mathscr{E}) \subseteqq T[X]$. This completes the proof of Theorem 13.

THEOREM 14. *Let $\mathscr{P}$ be a complete Boolean algebra of $\mathscr{F}$-projections on a Banach space that is reflexive and smooth. Then the weakly closed algebra $\mathscr{W}(\mathscr{P})$ of operators on X generated by $\mathscr{P}$ is equal to its second commutant.*

Proof. Bade [1] shows that if $\mathscr{P}$ is complete, then $\mathscr{W}(\mathscr{P})$ is the uniformly closed algebra of operators generated by $\mathscr{P}$ and it consists, furthermore, of exactly those (bounded linear) operators of X which leave invariant every closed linear manifold invariant under $\mathscr{P}$.

Suppose A is in the second commutant of $\mathscr{W}(\mathscr{P})$. For each x in X, let T^x denote the norm one projection whose range is $S_x = S(x; \mathscr{P})$. Then T^x commutes with $\mathscr{W}(\mathscr{P})$ so that $AT^x = T^x A$ for all x in X. From this, we have that A leaves each S_x invariant: $AS_x = AT^x X = T^x AX \subseteqq T^x X = S_x$. If M is a closed subspace left invariant under $\mathscr{P}$, then $S_m \subseteqq M$ for all m in M; whence, $A(m) \in AS_m \subseteqq S_m \subseteqq M$ for each m in M. Therefore, A leaves M invariant. Therefore, $A \in \mathscr{W}(\mathscr{P})$.

4. A class of examples. Let (S, Σ, μ) be a measure space with the property FSP (a measurable set of infinite measure contains a measurable subset of finite positive measure). This condition is discussed in [9]. We consider an Orlicz space L_M over (S, Σ, μ) where the complimentary Young's functions M and N are normalized ($M(1) + N(1) = 1$), satisfy Δ_2 conditions, and have continuous, strictly increasing derivatives denoted m and n, respectively. Then L_M is reflexive and [9; Corollary 2.1] the Luxemberg norms in both L_M and L_N are strongly differentiable. Furthermore, the weak derivative of a norm 1 function f_0 in L_M is given by $f \to \int f m(f_0) d\mu$.

LEMMA 15. *If $0 \leqq f \in L_M$, then $m\left(\frac{f(x)}{\|f\|}\right) = \frac{m(f(x))}{\|mf\|}$ for almost* *all* $x \in S$.

Proof. If $h = \alpha g$ for $\alpha \geqq 0$ and if $h, g \geqq 0$ a.e., we have equality for h and $m(g)$ in Holder's inequality: $\|h\|\,\|mg\| = \int hm(g)d\mu$. Then $\int fm\Big(\frac{f}{\|f\|}\Big)d\mu = \|f\| = \int f\Big(\frac{m(f)}{\|m(f)\|}\Big)d\mu$ so $m(f/\|f\|)$ and $m(f)/\|mf\|$ are normers for f. Since L_M is smooth, normers are unique.

LEMMA 16. *Assume the existence of sets of arbitrarily small positive measure. If $f, g \in L_M$ with $0 < \|f\| < \|g\|$, then $0 < \|mf\| < \|mg\|$.*

Proof. Set $K = \|g\|/\|f\| > 1$. Choose $x \in S$ such that $0 < m(g(x))/\|m(g)\| = m(g(x))/\|g\|)$. Set $a = |g(x)|/K > 0$. For any measurable set E, let f_E be the function constant on E at the value a, and agreeing with $|f|$ outside of E. By diminishing the measure of E, the function f_E may be brought in the norm of L_M as close to $|f|$ as desired. Furthermore, $\|m(Kf_E)\| - \|mf\|$ approaches $\|m(Kf)\| - \|mf\| > 0$ as E decreases. It is therefore, possible to choose a set E of positive measure so small that

$$m(g(x)/\|g\|)(\|f\|/\|f_E\|)\|m(Kf_E)\| > m(g(x)/\|g\|)\|mf\| .$$

Select $y \in E$ such that $m(Kf_E(y)) = m(Kf_E(y)/\|(Kf_E\|))\|m(Kf_E)\|$. Computing, we have

$$\begin{aligned} &m(g(x)/\|g\|)\|mg\| = m(g(x)) = m(Ka) = m(Kf_E(y)) \\ &= m(f_E(y)/\|f_E\|)\|m(Kf_E)\| = m(a/\|f_E\|)\|m(Kf_E)\| \\ &= m((g(x)/\|g\|)(\|f\|/\|f_E\|))\|m(Kf_E)\| > m(g(x)/\|g\|)\|mf\| . \end{aligned}$$

Cancelling $m(g(x)/\|g\|)$ finishes the argument.

Perhaps Lemma 16 is true without restrictions on the measure space. We have not settled this.

Define $\mathscr{F}(\lambda) = \|f\|\,\|mf\| = \int |f|\, m(f)d\mu$ where f is any function in L_M of norm λ. From Lemma 16, it is clear that $\mathscr{F}$ is well defined and strictly increasing. To show continuity, let E be any set of finite positive measure and a $(\lambda) = \lambda/\|\chi_E\|$. Then $a(\lambda)$ is continuous and

$$\mathscr{F}(\lambda) = \int a(\lambda)\chi_E m(a(\lambda)\chi_E)d\mu = \int a(\lambda)m(a(\lambda))\chi_E d\mu = a(\lambda)m(a(\lambda))\mu E ,$$

a continuous function.

Each measurable set E gives rise to the characteristic projection $f \rightarrow \chi_E f$.

LEMMA 17. *Every characteristic projection is an $\mathscr{F}$-projection.*

Proof.

$$\begin{aligned}\mathscr{F}(\|f\|) &= \int fm(f)d\mu = \int_E fm(f)d\mu + \int_{S\setminus E} fm(f)d\mu \\ &= \int (\chi_E f)m(\chi_E f)d\mu + \int (\chi_{S\setminus E} f)m(\chi_{S\setminus E} f)d\mu \\ &= \mathscr{F}(\|\chi_E f\|) + \mathscr{F}(\|\chi_{S\setminus E} f\|) .\end{aligned}$$

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