COUNTABLE BOOLEAN ALGEBRAS AS SUBALGEBRAS AND HOMOMORPHS

T. E. CRAMER

The problem of classifying all countable Boolean algebras appears to be impossible to solve. This paper considers the following problem. Given a class $\mathscr{C}$ of countable Boolean algebras, which is closed under isomorphisms, characterize the classes of

(i) all Boolean algebras which have subalgebras in $\mathscr{C}$;
(ii) all subalgebras af members of $\mathscr{C}$;
(iii) all homomorphs of members of $\mathscr{C}$;
(iv) all Boolean algebras which have homomorphs in $\mathscr{C}$.

Definitive characterizations are obtained for the first three classes (Theorems 7, 8, and 9), and a representation of the last class is obtained when $\mathscr{C}$ is the class of all countable Boolean algebras (Theorem 11).

Given a Boolean algebra A, let X be the corresponding Boolean space. Define inductively: $D_0(X)$ as X; $D_1(X)$ as the complement of the isolated points in X; for any ordinal α, $D_{\alpha+1}(X)$ as $D_1(D_\alpha(X))$; and for a limit ordinal α, $D_\alpha(X)$ as $\cap \{D_\beta(X): \beta < \alpha\}$. Then $D_\alpha(X)$ is a closed subspace of X for each ordinal α. The Boolean algebra A is said to be *superatomic* if $D_\alpha(X) = \varnothing$, for some ordinal α. If γ is the last ordinal for which $D_\gamma(X) = \varnothing$, the *cardinal sequence* $\Gamma(A)$ of the superatomic Boolean algebra A is defined in [1] as the sequence of order type γ whose α-term is the cardinality of the set of isolated points of $D_\alpha(X)$, $\alpha < \gamma$. Note that each term of $\Gamma(A)$ is infinite except for the $\gamma - 1$ term, which must be finite.

If $\mathscr{S}$ is the class of all superatomic Boolean algebras, the class $\{\Gamma(A): A \in \mathscr{S}\}$ may be partially ordered by setting $\Gamma(A) \leq \Gamma(B)$ when the order type of $\Gamma(A)$ is not larger than that of $\Gamma(B)$, and when the α-term of $\Gamma(A)$ is not larger than the α-term of $\Gamma(B)$. It is shown in [4] that every countable superatomic Boolean algebra has as its Boolean space the ordinal $\omega^\alpha n + 1$ endowed with the order topology, for some countable ordinal α and natural numbers n. Thus the relation "$\leq$" well orders the class of cardinal sequences of countable superatomic Boolean algebras.

LEMMA 1. *If X and Y are Boolean spaces, and θ is a continuous function from X onto Y, then $\theta[D_\alpha(X)]$ contains $D_\alpha(Y)$, for each ordinal α.*

Proof. Assume that $\alpha = 1$, and that y is an element of $Y -$

$\theta[D_1(X)]$. Then $X - \theta^{-1}(y)$ is closed because $\theta^{-1}(y)$ is the union of isolated points. Therefore, $\theta[X - \theta^{-1}(y)] = Y - \{y\}$ must be closed, so y is isolated and does not belong to $D_1(Y)$.

Now assume that $\theta[D_\alpha(X)]$ contains $D_\alpha(Y)$, for some ordinal α. Then $\theta[D_{\alpha+1}(X)] = \theta[D_1(D_\alpha(X))] \supseteq D_1(D_\alpha(Y)) = D_{\alpha+1}(Y)$.

Finally, assume $\theta[D_\alpha(X)] \supseteq D_\alpha(Y)$ for each $\alpha < \beta$, β some limit ordinal. Let y be a element of $D_\beta(Y)$, and define $K_\alpha = D_\alpha(X) \cap [\theta^{-1}(y)]$. By hypothesis, K_α is closed and nonempty for each $\alpha < \beta$. Since X is compact, $\cap \{K_\alpha : \alpha < \beta\} \neq \varnothing$, so let x be a point in this intersection. Then x is in $\cap \{D_\alpha(X) : \alpha < \beta\}$ and is in $\theta^{-1}(y)$, so $\theta[D_\beta(X)] \supseteq D_\beta(Y)$.

LEMMA 2. *If X is a Boolean space and Y is a clopen subspace of Y, then $D_\alpha(X) \cap Y = D_\alpha(Y)$ for every ordinal α.*

Proof. By induction.

THEOREM 3. *If A and B are superatomic Boolean algebras, A is countable, and $\Gamma(A) \leq \Gamma(B)$, then there is a monomorphism from A to B.*

Proof. Since A is a countable superatomic Boolean algebra, its Boolean space can be represented as $\omega^\alpha n + 1$ with the order topology, for some countable ordinal α, and some natural number n. Let Y be the Boolean space of B. The dual statement to the theorem will be proved by constructing a continuous mapping of Y onto $\omega^\alpha n + 1$. Two topological observations will be needed.

(1) Suppose that $\{(Y_\beta, X_\beta) : \beta < \alpha\}$ is a set of pairs of topological spaces. If there are continuous functions from Y_β to X_β, for all $\beta < \alpha$, then there is a continuous function from the disjoint union $\cup \{Y_\beta : \beta < \alpha\}$ to the disjoint union $\cup \{X_\beta : \beta < \alpha\}$.

(2) Suppose that Y is a topological space with an open subspace U. If f is a continuous function from U to a topological space X, and $f^{-1}(V)$ is closed in Y whenever V is closed and compact in X, then there is a continuous extension g of f from Y to the one-point compactification of X. Define $g(y) = f(y)$ when $y \in U$, and $g(y) = x_0$ when $y \in Y - U$, where x_0 is the compactification point.

The proof proceeds by inducting first on α and then on n. If $a = 0$ then the Boolean space of A is finite and has the discrete topology. Since $\Gamma(A) \leq \Gamma(B)$, the cardinality of Y is at least as large as that of the Boolean space of A, so an application of (1) produces the desired function between these spaces.

If $\alpha = \beta + 1$ for some ordinal β, and $n = 1$, then $D_\beta(Y)$ is in-

finite. Pick a sequence $\{y_n : n < \omega\}$ of distinct points in $D_\beta(Y)$, and choose disjoint clopen sets $\{Y_n : n < \omega\}$ in Y such that $y_n \in Y_n$, for each $n < \omega$. Lemma 2 implies that $D_\beta(Y_n) \neq \varnothing$ for each $n < \omega$, when Y_n is considered as a Boolean space. Write $\omega^{\beta+1} + 1$ as

$$[0, \omega^\beta] \cup [\omega^\beta + 1, \omega^\beta 2] \cup \cdots \cup \{\omega^{\beta+1}\} \; .$$

Each subset $[\omega^\beta n + 1, \omega^\beta(n + 1)]$ is homeomorphic to $\omega^\beta + 1$, so the induction hypothesis insures a continuous mapping of Y_n onto

$$[\omega^\beta n + 1, \omega^\beta(n + 1)], n < \omega \; .$$

Apply (1) and then (2).

Assume now that α is a limit ordinal and $n = 1$. Choose a strictly increasing sequence $\alpha(0)$, $\alpha(1)$, $\alpha(2)$, $\cdots$ converging to α. Pick points y_n in $D_{\alpha(n)}(Y) - D_{\alpha(n+1)}(Y)$, $n < \omega$, and let $\{Y_n : n < \omega\}$ be a set of disjoint clopen sets in Y such that $y_n \in Y_n$, $n < \omega$. Represent $\omega^\alpha + 1$ as $[0, \omega^{\alpha(0)}] \cup [\omega^{\alpha(0)} + 1, \omega^{\alpha(1)}] \cup \cdots \cup \{\omega^\alpha\}$. As before, there must be continuous functions from Y_n onto $[\omega^{\alpha(n)} + 1, \omega^{\alpha(n+1)}]$, and a continuous extension from Y onto $\omega^\alpha + 1$.

Finally, if $n > 1$, $D_\alpha(Y)$ contains at least n points. Pick a clopen set Y_0 with complement Y_1, such that $Y_0 \cap D_\alpha(Y)$ contains at least $n - 1$ points, and $Y_1 \cap D_\alpha(Y) \neq \varnothing$. By the induction hypothesis there are continuous functions mapping Y_0 onto $[0, \omega^\alpha(n - 1)]$, and Y_1 onto $[\omega^\alpha(n - 1) + 1, \omega^\alpha n + 1]$. Apply (1) to prove the existence of a continuous function from Y onto $\omega^\alpha n + 1$.

THEOREM 4. *If A and B are countable superatomic Boolean algebras, then the following are equivalent*:

   (i) $\Gamma(A) \leqq \Gamma(B)$;
   (ii) *there is a monomorphism from A to B*;
  (iii) *there is an epimorphism from B to A*;
  (iv) *A is a retract of B*.

Proof. Let $X = \omega^\alpha m + 1$ be the Boolean space of A, and $Y = \omega^\beta n + 1$ that of B, both with the order topology, α and β countable ordinals, m and n natural numbers. Part (i) implies (ii) by Theorem 3. Assuming (ii), there is a continuous function θ from Y onto X; $\theta[D_\beta(Y)]$ is finite and contains $D_\beta(X)$ by Lemma 1, so (ii) implies (i). Part (i) implies (iii) because X can be identified with the initial segment of Y, and (iii) implies (i) by Lemma 2. Finally, (iii) is equivalent to (iv) because A is countable and therefore projective [3].

If A and B are superatomic Boolean algebras and A is countable, then an epimorphism from B to A would insure the existence of a monomorphism from A to B. However, there are superatomic Boolean

algebras A and B, A countable, and $\Gamma(A) \leqq \Gamma(B)$, for which there is no epimorphism from B to A.

EXAMPLE 5. Let N be the set of natural numbers with the discrete topology. Pick an uncountable family of subsets of N such that each pair intersect in at most a finite number of points. (See [5].) Extend this family to a family $\mathscr{F} = \{F_i: i \in I\}$, indexed by some set I, maximal with respect to the property that every pair has a finite intersection. Let $J \subseteq I$ be the set of indices for which F_i is infinite. Adjoin to N a set $\{x_i: i \in J\}$ of new points, and take the family of all sets $U \cup \{x_i\}$, where U is a co-finite subset of F_i, as open neighborhoods of x_i, $i \in J$. Let X be the one-point compactification, by the point x_0, of the resulting topological space. It is easily seen that X is Hausdorff, and has a basis of clopen sets. Moreover, $D_1(X)$ is the one-point compactification of the set of its isolated points $\{x_i: i \in J\}$, so the Boolean algebra associated with X is superatomic with cardinal sequence $\aleph_0$, $|J|$, 1.

Suppose that there is a homeomorphism θ of $\omega^2 + 1$ onto Y, Y a closed subspace of X. Since $D_2(Y) \subseteq D_2(X)$, $\theta(\{\omega^2\}) = x_0$, and since $D_1(Y) \subseteq D_1(X)$, $\{\theta(\omega \cdot n): n < \omega\}$ is a countable infinite subset of $\{x_i: i \in J\}$. There are disjoint, clopen subsets U_n, $n < \omega$, in X such that $U_n \cap D_1(X) = \{\theta(\omega \cdot n)\}$, and $U_n \cap Y = [U_n \cap Y \cap N] \cup \{\theta(\omega \cdot n)\}$ is infinite for all $n < \omega$. Pick some point $y_n \in U_n \cap Y \cap N$ for each $n < \omega$. By the maximality assumption, there is a set $F_i \in \mathscr{F}$ for which $F_i \cap \{y_n: n < \omega\}$ is infinite. Then $x_i \notin Y$, but x_i is an accumulation point of Y, so Y was not closed in X.

TERMINOLOGY. For the rest of the paper, $\mathscr{S}$ will designate the class of superatomic Boolean algebras. The infinite countable free Boolean algebra will be denoted F_0. In Theorems 7, 8, 9, and 10, $\mathscr{C}$ will be a nonempty class of infinite, countable Boolean algebras, closed under isomorphisms. The finite Boolean algebras will not be considered, because the situation in this case is well known. A finite Boolean algebra is a retract of every Boolean algebra of larger cardinality.

LEMMA 6. *If C is a countable, infinite, nonsuperatomic Boolean algebra, then every countable Boolean algebra A is a retract of C.*

Proof. Because C is nonsuperatomic, C has an atomless homomorph. Every countable, infinite, atomless Boolean algebra is isomorphic to F_0. (See [2].) Since every countable Boolean algebra A is a homomorph of F_0, A is a retract of C.

Theorems 7 through 10 follow readily from Theorems 3 and 4, Lemma 6, and the facts: (A) Every countable Boolean algebra is projective. (B) A Boolean algebra is superatomic if and only if it has no subalgebra isomorphic to F_0.

THEOREM 7. *Let $\mathscr{A}$ be the class of Boolean algebras which have a subalgebra in $\mathscr{C}$.*

(i) *If $\mathscr{C}$ contatins no superatomic Boolean algebras then $\mathscr{A}$ is the class of all nonsuperatomic Boolean algebras.*

(ii) *If C_0 is the superatomic Boolean algebra in $\mathscr{C}$ with the smallest cardinal sequence, then $\mathscr{A}$ is the class of all Boolean algebras except $\{A \in \mathscr{S} : \Gamma(A) \nleqq \Gamma(C_0)\}$.*

THEOREM 8. *Let $\mathscr{A}$ be the class of subalgebras of members of $\mathscr{C}$.*

(i) *If $\mathscr{C}$ contains a nonsuperatomic Boolean algebra C_0, then $\mathscr{A}$ is the class of all countable Boolean algebras.*

(ii) *If $\mathscr{C}$ is a class of superatomic Boolean algebras, then $\mathscr{A}$ is the class of countable superatomic Boolean algebras A for which $\Gamma(A) \leqq \Gamma(C)$ for some $C \in \mathscr{C}$.*

THEOREM 9. *The class of all homomorphs of members of $\mathscr{C}$ is just the class of all subalgebras of members of $\mathscr{C}$.*

THEOREM 10. *Let $\mathscr{A}$ be the class of Boolean algebras which have a homomorph in $\mathscr{C}$.*

(i) *If $\mathscr{C}$ contains no superatomic Boolean algebras, then $\mathscr{A}$ is the class of Boolean algebras with a countable infinite, free retract.*

(ii) *If C_0 is the superatomic Boolean algebra in $\mathscr{C}$ with the least cardinal sequence, then $\mathscr{A}$ is the class of Boolean algebras which have C_0 as a retract.*

The general problem of determining when a Boolean algebra is a retract of another seems to be quite difficult. The situation is not simple even in the case of superatomic Boolean algebras, as was seen in Example 5. Some results are obtained when attention is restricted to the field of all finite and co-finite subsets of a countable infinite set, however.

Let P_0 denote the power set on a countable, infinite set, and let T_0 be the quotient of P_0 by the ideal generated by its atoms. Let S_0 be the field of all finite and co-finite subsets of a countable, infinite set. If B is a subalgebra of a Boolean algebra A, a *simple extension of B in A* is a subalgebra of A generated by the set $\{a\} \cup B$, where a is some element in A.

THEOREM 11. *A Boolean algebra A has no countable, infinite homomorph if and only if for every infinite homomorph B of A, there is an epimorphism from B to a subalgebra C of T_0 such that every simple extension of C in T_0 is atomless.*

Proof. By Theorem 10, a Boolean algebra A has no countable, infinite homomorph if and only if S_0 is not a homomorph of A.

Suppose that S_0 is not a homomorph of A, and that B is an infinite homomorph. Pick a set $\{b_n : n < \omega\}$ of disjoint elements in B. Let D be the quotient of B by an ideal maximal with respect to not containing any element b_n, $n < \omega$. Then D is an infinite atomic Boolean algebra with a countable set X of atoms. The completion P of D is isomorphic to the power set on X, and we shall consider D as a subalgebra of P. Let C be the quotient $D/(X)$ of D by the ideal generated by X, and notice that C can be embedded in $P/(X)$. This latter quotient is isomorphic to T_0.

Suppose a is an atom in a simple extension of C in T_0. Then every member of C either contains $\{a\}$ or is disjoint from $\{a\}$. This can happen in the quotient algebras only when there is an infinite set Y of atoms in D, and every element of D contains either a finite or co-finite subset of Y. Thus the quotient of D by the ideal of all elements disjoint from the members of Y is isomorphic to S_0, but also is a homomorph of A.

Conversely, if A has an infinite, countable homomorph, then S_0 is also a homomorph. Every homomorph of S_0 is atomic, so there can be no epimorphism from S_0 to a subalgebra of T_0 with every simple extension atomless.

REFERENCES

1. G. W. Day, *Superatomic Boolean algebras*, Pacific J. Math. **23** (1967), 479–489.
2. Ph. Dwinger, *Introduction to Boolean algebras*, Wurzburg, 1961.
3. P. R. Halmos, *Injective and projective Boolean algebras*, Proc. Sympos. Pure Math. **2** (1961), 114–122.
4. R. D. Mayer and R. S. Pierce, *Boolean algebras with ordered bases*, Pacific J. Math. **10** (1960), 925–942.
5. W. Sierpinski, *Cardinal and Ordinal Numbers*, Warsaw, 1958.

UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BRITISH COLUMBIA