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# ALMOST SMOOTH PERTURBATIONS OF SELF-ADJOINT OPERATORS

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**Assume $H^0 \in \mathscr{C}(\mathfrak{H})$ is a self-adjoint operator with spectrum on $[o, \infty)$ and that $E^0(\Delta) \in \mathscr{B}(\mathfrak{H})$ is the spectral measure determined by $H^0$, $\Delta \subset [0, \infty)$. Let $H^1 = H^0 + V$ where $V = B \cdot A$ and $A, B \in \mathscr{B}(\mathfrak{H})$ are commuting self-adjoint operators. In this paper T. Kato's concept of smooth perturbations is generalized in the following way: $H^1$ is said to be an almost smooth perturbation of $H^0$, except at $1 = 0$, if $A, B$ are smooth with respect to $H^0E^0(\Delta_m)$ for all intervals $\Delta_m = (1/m, \infty)$, $m \geqq 1$. It is proved that the time independent wave operators corresponding to $H^0, H^1$ exist when the assumption that $H^1$ is smooth with respect to $H^0$ is replaced by the assumption that $H^1$ is almost smooth with respect to $H^0$.**

The concept of smooth perturbations was introduced by T. Kato in [2]. The importance of the generalization given here is that it allows one to apply the theory developed in [2] to certain one dimensional differential operators which are almost smooth but not smooth. Examples of some almost smooth ordinary differential operators are given below in § 3.

2. **The wave operators.** Let $\Omega_\pm$ denote the upper and lower complex plane, with the reals excluded, and let $f$ be a function on $\Omega_\pm \times \mathfrak{H}$ into $\mathfrak{H}$. Such a function $f$ is said to be in the Hardy class $H_2((-\infty, \infty): \mathfrak{H})$ if and only if $f$ is analytic in $\lambda$ for all $\lambda \in \Omega_\pm$ and for all $u \in \mathfrak{H}$, $\delta > 0$, $\int_{-\infty}^{\infty} \|f(1 \pm i\delta; u)\|^2 d1 \leqq P\|u\|^2$ for some $P > 0$ independent of $u$ and $\delta$. An operator $A \in \mathscr{B}(\mathfrak{H})$ is said to be smooth with respect to $H^0$ if and only if the function $f$ defined by

$$(\lambda, u) \mapsto A(H^0 - \lambda I)^{-1}u = AR^0(\lambda)u \text{ is in } H_2((-\infty, \infty): \mathfrak{H})$$

[2, p. 260].[1] Now we shall make the following assumptions regarding $H^0, A, B$:

(i) For some $N$, $\|BR^0(\lambda)E^0(\Delta_m)A\| \leqq K < 1$ for $m \geqq N$ and for all $\lambda$ not real positive or zero.

(ii) $H^1 = H^0 + BA$ is an almost smooth perturbation of $H^0$. It will be shown below that these two assumptions insure the existence of the wave operators in the time independent form. With additional assumptions, one may also show that these operators coincide with

[1] Actually T. Kato defines smoothness for more general operators than those considered in this paper.

the wave operators defined in the time dependent manner. For this purpose we shall also assume:

(iii) For some $\lambda_0$, $\mathrm{Im}(\lambda_0) \neq 0$, the operator $|V|^{1/2}R^0(\lambda_0)$ is an integral operator with kernel in the Schmidt class.

Let $R^p(\lambda) \in \mathscr{B}(\mathfrak{H})$ denote the resolvent operators corresponding to the operators $H^p$, $p = 0, 1$ (in formulas involving both $H^0$, $H^1$ we shall use $p, q = 0, 1, p + q = 1$).

LEMMA 1 (*T. Kato*). *Let $A, B$ be smooth with respect to $H^0$ and let* (i) *hold. Then for, $1 \in [0, \infty)$, $\delta > 0$, $p = 0, 1$ $R^1(1 \pm i\delta)$ is defined in terms of $R^0(1 \pm i\delta)$ by*

$$\begin{aligned} R^1(1 \pm i\delta) &= R^0(1 \pm i\delta) \\ &\quad - R^0(1 \pm i\delta)B(1 + Q(1 \pm i\delta))^{-1}AR^0(1 \pm i\delta) \end{aligned} \tag{2.1}$$

*where $Q(\lambda) = AR^0(\lambda)B$.*

*Proof.* The proof is given in [2, p. 263]. Note that formula (2.1) is essentially the Neumann series for the resolvent since by assumption (i) $\sum_{v=0}^{\infty}(-1)^v Q^v$ is norm convergent and

$$(1 + Q)^{-1} = \sum_{v=0}^{\infty}(-1)^v Q^v .$$

LEMMA 2 (*T. Kato*). *Let $A, B$ be smooth with respect to $H^0$ and let* (i) *hold. Then for $u \in \mathfrak{H}$, $1 \in [0, \infty)$, $\delta > 0$, $p = 0, 1$:*

1. *The vectors $AR^p(1 \pm i\delta)u$, $BR^p(1 \pm i\delta)u$ have limits along the reals $\delta \to 0$ in $\mathfrak{H}$.*

2. *If $AR^p(1 \pm i0)u$, $BR^p(1 \pm i0)u$ denote the limits $AR^p(1 \pm i0)u = \lim_{\delta\to 0} AR^p(1 \pm i\delta)u$, $BR^p(1 \pm i0)u = \lim_{\delta\to 0} BR^p(1 \pm i\delta)u$ then $AR^p(1 \pm i0)u$, $BR^p(1 \pm i0)u \in L_2([0, \infty): \mathfrak{H})$.*

3. *If $Q = \sup_{\lambda,m} \|BR^0(\lambda)E^0(\Delta_m)A\|$, $\mathrm{Im}(\lambda) \neq 0$, $m \geqq N$, then*

$$\|AR^1(1 \pm i0)u\| \leqq (1 - \|Q\|)^{-1}\|AR^0(1 \pm i0)u\|$$

*and*

$$\|BR^1(1 \pm i0)u\| \leqq (1 - \|Q\|)^{-1}\|BR^0(1 \pm i0)u\| .$$

*Proof.* [2, p. 264]. Note that in general the expression $R^p(1 \pm i0)u$ does not make sense. The uniform bound $\|Q\|$ exists by the principle of uniform boundedness.

THEOREM 1. *Let $A, B$ be smooth with respect to $H^0$ and let assumption* (i) *hold. Then the spectral measures $E^p(\Delta)$ corresponding to the operators $H^p$, $p = 0, 1$ are given by*

$$
\begin{aligned}
&(E^p(\Delta)u, v) = (E^q(\Delta)u, v) \\
&+ (-1)^p \frac{1}{\pi}\int_\Delta (\{R^p(1 \pm i0)AB\,\mathrm{Im}(R^q(1 \pm i0)) \\
&+ \mathrm{Im}(R^q(1 \pm i0)BAR^p(1 \mp i0)\}u, v)d1 \\
&+ \frac{1}{\pi}\int_\Delta (\{R^p(1 \pm i0)AB\,\mathrm{Im}(R^q(1 + i0))BAR^p(1 \mp i0)\}u, v)d1
\end{aligned} \tag{2.2}
$$

*where* $\mathrm{Im}(R^p(\lambda)) = (1/2i)(R^p(\lambda) - R^p(\bar{\lambda}))$, $u, v \in \mathfrak{H}$, $\Delta \subseteqq [0, \infty)$.

*Proof.* Lemma 2 implies that the terms in the integrand are well defined. The integrands are absolutely integrable functions on $[0, \infty)$ because of the hypothesis that $A, B$ are smooth with respect to $H^0$.

To see this consider the integral

$$\int_\Delta |R^p(1 + i0)BAR^q(1 - i0)u, v)|d1 . \tag{2.3}$$

By the Schwarz inequality (2.3) is dominated by

$$\int_\Delta \| AR^q(1 + i0)u\|^2 d1 \int_\Delta \| BR^p(1 - i0)v\|^2 d1 . \tag{2.4}$$

By Lemma 1,2 the product is less than

$$p^2(1 - \|Q\|)^{-1}\|u\|\|v\| .$$

Therefore

$$\int_\Delta |(R^p(1 - i0)BAR^q(1 + i0)u, v)|d1 \leqq p^2(1 - \|Q\|)^{-1}\|u\|\|v\| \tag{2.5}$$

for all $\Delta \subseteqq [0, \infty)$. The integral (2.3) corresponds to the first term in the integral on the right of (2.2). Bounds may be found for the integrals of the other terms of (2.2) similarly.

From the second resolvent equation $R^1(\lambda) = R^0(\lambda) - R^0(\lambda)BAR^1(\lambda)$ one may derive the identity

$$
\begin{aligned}
&\mathrm{Im}(R^p(1 \pm i\delta)) = \\
&(1 + (-1)^p R^p(1 \pm i\delta)AB)\mathrm{Im}(R^q(1 \pm i\delta)(1 + (-1)^p BAR^p(1 \mp i\delta)
\end{aligned} \tag{2.6}
$$

for all $1 \in [0, \infty)$, $\delta > 0$. Also it is known [2, p. 273] that

$$(E^p(\Delta)u, v) = \lim_{\delta \to 0} \frac{1}{\pi}\int_\Delta (\mathrm{Im}(R^p(1 \pm i\delta))u, v)d1 . \tag{2.7}$$

Employing (2.6), (2.7) and passing to the limit $\delta \to 0$ leads to (2.2).

THEOREM 2. *Let $A, B$ be smooth with respect to $H^0$ and let assumption* (i) *hold. Then the time independent wave operators $W_\pm^p$, $p = 0, 1$ are determined by*

$$(W_\pm^p u, v) = (u, v) + (-1)^p \frac{1}{\pi}\int_{0+}^{\infty} (A \operatorname{Im}(R^q(1 \pm i0))u, BR^p(1 \pm i0)v)dl \tag{2.8}$$

*for $u, v \in \mathfrak{H}$, $p + q = 1$.*

*Proof.* By Lemma 2 the integrand in (2.8) is well defined and it is absolutely convergent by the hypothesis that $A, B$ are smooth with respect to $H^0$. It is proved in [2] that $W_\pm^p$ are in general represented by

$$(W_\pm^p u, v) = (u, v)^\pm(-1)^p \frac{1}{2\pi i}\int_{-\infty}^{\infty} (AR^q(1 \pm i0)u, BR^p(1 \pm i0)v)dl\ . \tag{2.9}$$

The integral appearing on the right side of (2.9) determines a bounded operator $X_\pm^p$ and $(X_\pm^p u, v)$ may be written

$$(X_\pm^p u, v) = \lim_{\delta\to 0}(\pm)\frac{1}{2\pi i}\int_{-\infty}^{\infty} (R^p(1 \mp i\delta)BAR^q(1 \pm i\delta)u, v)dl\ . \tag{2.10}$$

Since we have assumed that the spectrum of $H^0$ lies on $[0, \infty)$ it follows using Lemma 1 that $R^p(\lambda)$, $p = 0, 1$ are regular in $\lambda$ if $\lambda$ is not real positive. Therefore the part of the integral (2.10) which is along the negative real axis may be deformed and

$$\begin{aligned}(X_\pm^p u, v) &= \lim_{\delta\to 0}\Big[(\pm)\frac{1}{2\pi i}\int_{0+}^{\infty} (R^p(1 \mp i\delta)BAR^q(1 \pm i\delta)u, v)dl \\ &\qquad (\pm)\frac{1}{2\pi i}\int_{0+}^{\infty} (R^p(1 \mp i\delta)BAR^q(1 \mp i\delta)u, v)dl\Big] \\ &= \lim_{\delta\to 0}\frac{1}{\pi}\int_{0+}^{\infty} (R^p(1 \mp i\delta)BA \operatorname{Im}(R^q(1 \pm i\delta)u, v)dl\ .\end{aligned} \tag{2.11}$$

Using (2.11) formula (2.9) reduces to (2.8).

COROLLARY. *Then wave operators are also given by*

$$W_\pm^p(u, v) = (u, v) + (-1)q \frac{1}{\pi}\int_{0+}^{\infty} (AR^q(1 \pm i0)u, B \operatorname{Im}(R^p(1 \pm i0))v)dl \tag{2.12}$$

*for $u, v \in \mathfrak{H}$, $p + q = 1$.*

*Proof.* The proof is similar to the proof of the theorem.

THEOREM 3. *Let $A, B$ be almost smooth with respect to $H^0$ and let assumption* (i) *hold. Then:*

1. *The equation*

$$(2.13)\qquad (W_{\pm}^{p}u, v) = (u, v) + (-1)^{p}\frac{1}{\pi}\int_{0+}^{\infty} (A\,\mathrm{Im}(R^{q}(1 \pm io))u, BR^{p}(1 \pm i0)v)d1$$

*defines operators $W_{\pm}^{p} \in \mathscr{B}(\mathfrak{H})$, $p = 0, 1$ such that $W_{\pm}^{p}W_{\pm}^{q} = 1$ and*

$$(2.14)\qquad W_{\pm}^{p}H^{q}W_{\pm}^{q} = H^{p}\,, \qquad p = 0, 1\,.$$

2. *If in addition assumption* (iii) *holds then $W_{\pm}^{p}$ coincide with the wave operators defined by the time dependent method.*

*Proof*. Consider a sequence of operators $H_n^0$ where $H_n^0 = H^0E^0(\Delta_n)$, $\Delta_n = (1/n, \infty)$, $n \geqq 1$. Let $H_n^1$ be the sequence such that $H_n^1 = H_n^0 + B\cdot A$, $n \geqq 1$. By hypothesis $A, B$ are smooth with respect to $H_n^0$ for all $n \geqq 1$. If $R_n^p(\lambda) = (H_n^p - \lambda I)^{-1}$ then by assumption (i) we may choose $N$ such that for $n \geqq N$, $\|BR_n^0(\lambda)A\| \leqq K < 1$ for all $\lambda$ which are not real positive or zero. Applying Theorem 2 the wave operators $W_{\pm}^{p}(n)$ corresponding to $H_n^p$ exist and are given in time independent form by

$$(2.15)\qquad (W_{\pm}^{p}(n)u, v) = (u, v) + (-1)^{p}\frac{1}{\pi}\int_{0+}^{\infty} (A\,\mathrm{Im}(R_n^{q}(1 \pm i0))u, BR_n^{p}(1 \pm i0)v)d1\,.$$

The operators $W_{\pm}^{p}(n)$ are in $\mathscr{B}(\mathfrak{H})$ and satisfy $W_{\pm}^{p}(n)W_{\pm}^{q}(n) = I$, $p + q = 1$ and

$$(2.16)\qquad W_{\pm}^{p}(n)R_n^{q}(\lambda)W_{\pm}^{q}(n) = R_{\pm}^{p}(\lambda)$$

for all $\lambda$, $\mathrm{Im}(\lambda) \neq 0$. Also if $E_n^p(\Delta)$, $\Delta \subseteqq [0, \infty)$ are the spectral measures of $H_n^p$ then $E_n^p(\Delta)W_{\pm}^{p}(n) = W_{\pm}^{p}(n)E_n^q(\Delta)$, $\Delta \subseteqq [0, \infty)$, which follows from (2.16). The operators $W_{\pm}^{p}(n)$ agree with the wave operators defined in terms of $H_n^p$ in the time dependent manner [2, p. 271] and they satisfy $|(W_{\pm}^{p}(n)u, v)| \leqq \|E^p(\Delta_n)v\|\|E^q(\Delta_n)u\|$. Formula (2.15) may be written

$$(2.17)\qquad (W_{\pm}^{p}(n)u, v) = (u, v) + (-1)^{p}\int_{0+}^{\infty} (A\,\mathrm{Im}(R^{q}(1 \pm i0))E^{q}(\Delta_n)u, BR^{p}(1 \pm i0)E^{p}(\Delta_n)v)d1\,.$$

Now consider the expression $Z_n(u, v)$ defined by

$$(2.18)\qquad Z_n(u, v) = (E^{q}(\Delta_n)u, v) + (-1)^{p}\frac{1}{\pi}\int_{1/n+}^{\infty} (A\,\mathrm{Im}(R^{q}(1 \pm i0))u, BR^{p}(1 \pm i0)v)d1\,.$$

To see that the integral in (2.18) is well defined write

$$R^p(\lambda) = S^p_m(\lambda) + R^p_m(\lambda) \text{ where } R^p_m(\lambda) = R^p(\lambda)E^p(\Delta_m)$$

and

$$S^p_m(\lambda) = R^p(\lambda)(I - E^p(\Delta_m)) = \int_0^{1/m} (1_1 - \lambda)^{-1} dE^p(1_1) .$$

The operator $S^p_m(\lambda)$ is regular in $\lambda$ for $\lambda \in (1/n, \infty)$, $m > n$. By hypothesis $A, B$ are smooth with respect to $H^0_m$. Applying Lemma 2,

$$\lim_{\delta\to 0} AR^p(1 \pm i\delta)u = AS^p_m(1)u + AR^p_m(1 \pm i0)u \in \mathfrak{H} ,$$

$$\lim_{\delta\to 0} BR^p(1 \pm i\delta)u = BS^p_m(1)u + BR^p_m(1 \pm i0)u \in \mathfrak{H}, 1 \in (1/n, \infty)$$

and the integrals

$$\int_{1/n}^\infty \|AS^q_m(1)u\|^2 d1, \int_{1/n}^\infty \|AR^q_m(1 \pm io)u\|^2 d1,$$
$$\int_{1/n}^\infty \|BS^p_m(1)u\|^2 d1, \int_{1/n}^\infty \|BR^p_m(1 \pm i0)u\|^2 d1$$

are convergent. Therefore the integral in (2.18) is absolutely convergent by the Schwarz inequality. Writing $Z_n(u, v)$ in the form

$$(2.19)\quad \begin{aligned} Z_n(u, v) = \lim_{\delta\to 0} \frac{1}{\pi}\int_{1/n+}^\infty \\ (((1 + (-1)^p R^p(1 \pm i\delta)BA) \operatorname{Im}(R^q(1 \pm i\delta))u, v)d1 \end{aligned}$$

by (2.5) and applying (2.6) one obtains

$$(2.20)\quad \begin{aligned} Z_n(u, v) = \lim_{\delta\to 0} \int_{1/n+}^\infty \\ ((1 + (-1)^q BAR^q(1 \pm i\delta))u, \operatorname{Im}(R^p(1 \mp i\delta))v)d1 . \end{aligned}$$

This implies, again by (2.5),

$$(2.21)\quad \begin{aligned} Z_n(u, v) = (u, E^p(\Delta_n)v) + (-1)^q \frac{1}{\pi}\int_{1/n+}^\infty \\ (BAR^q(1 \pm i0)u, \operatorname{Im}(R^p(1 \mp i0))v)d1 \end{aligned}$$

Now because of the regularity of $S^p_n(1)$ for $1 \in (1/n, \infty)$ and $R^p_n(1)$ for $1 \in (0, 1/n)$ we have $\operatorname{Im}(R^p_n(1 \pm i0))u = 0$ for $1 \in (0, 1/n)$ and

$$\operatorname{Im}(R^p_n(1 \pm i0))u = \operatorname{Im}(R^p(1 \pm i0))u ,$$

for $1 \in (1/n, \infty)$. Using these relations it follows from (2.17), (2.18), (2.20) that $Z_n(u, v) = Z_n(E^q(\Delta_n)u, v) = Z_n(u, E^p(\Delta_n)v)$ and also

$$Z_n(u, v) = (W^p_\pm(n)E^q(\Delta_n)u, E^p(\Delta_n)v) = (W^p_\pm(n)u, v) .$$

Since

$$Z_n(u, v) = (W_\pm^p(n)u, v), |Z_n(u, v)| \leqq \|E^p(\Delta_n)v\| \|E^q(\Delta_n)u\| ,$$

the sequence $Z_n(u, v)$ converges to the right side of (2.13) and defines an operator $W_\pm^p \in \mathscr{B}(\mathfrak{H})$. The operators $R_n^p(\lambda)$ converge strongly to $R^p(\lambda)$, $\mathrm{Im}(\lambda) \neq 0$, the operators $W_\pm^p(n)$ converge weakly to $W_\pm^p$, $n \to \infty$, and $W_\pm^p(n) W_\pm^p = I$, $W_\pm^p(n) R_n^q(\lambda) W_\pm^q(n) = R_n^p(\lambda)$, $n \geqq 1$. From these relations it follows that the first part of the conclusion of the theorem is valid.

If the hypothesis of part 2 of the theorem holds then the wave operators corresponding to $H^p$ exist as defined in the time dependent manner [1, p. 546]. Let us denote these wave operators by $\hat{W}_\pm^p$. It is easily seen that $W_\pm^p(n)$ converge strongly to $\hat{W}_\pm^p$. Since $W_\pm^p(n)$ also converge weakly to $W_\pm^p$ it must be that $W_\pm^p = \hat{W}_\pm^p$ and the operators defined in the two different ways coincide.

COROLLARY. *The conclusion of Theorem* 1 *holds if the assumption that $H^1$ is a smooth perturbation of $H^0$ is replaced by the assumption that $H^1$ is an almost smooth perturbation of $H^0$.*

The proof proceeds along the same lines as the proof of the theorem.

**3. Application to ordinary differential operators.** To apply Theorem 3 consider a self-adjoint operator $H^0$ on $L_2(-\infty, \infty)$ which is determined by the formal ordinary differential operator

$$L_0 = (-1)^v (d/dx)^n, n = 2v ,$$

defined on $(-\infty, \infty)$. The resolvent $R^0(\lambda) = (H^0 - \lambda I)^{-1}$ may be explicitly calculated. Let $\lambda$, $w$ be complex variables defined by

$$\lambda = re^{i\theta}, w = r^{1/n} \exp(i\theta/n), r \geqq 0, 0 \leqq \theta < 2\pi ,$$

and define functions $s_j^0(x, \lambda) = \exp(e_j w x)$, $j = 1, \cdots, n$ where $e_j$, $j = 1, \cdots, n$ are complex roots of unity with increasing argument

$$-\frac{\pi}{2} \leqq \arg e_1 < \cdots < \arg e_n < \frac{3\pi}{2} . \tag{3.1}$$

The functions $s_j^0(x, \lambda)$, $j = 1, \cdots, n$ form a fundamental set of solutions of the equation $L_0 y = \lambda y$, $-\infty < x < \infty$. The resolvent $R^0(\lambda)$ is an integral operator whose kernel is the Green's function

$$G^0(x, y: \lambda) = \frac{1}{nw^{n-1}} \sum_{k=v+1}^{n} e_k s_k^0(x, \lambda) s_{k-v}^0(y, \lambda) \tag{3.2}$$

for $x \geqq y$ with variables $x, y$ interchanged if $x < y$. The kernel of the spectral measure $E^0(\Delta)$ associated with $H^0$ is given by

$$\mathscr{E}^0(x, y\colon \Delta) = \lim_{\delta \to 0} \frac{1}{\pi} \int_\Delta \operatorname{Im}(G^0(x, y\colon 1 + i\delta)d1 .$$

More precisely and explicitly with $\Delta_1 = (\Delta)^{1/n}$, $s^n = 1$

$$\begin{aligned}E^0(\Delta)u = \frac{1}{2\pi i}\int_{\Delta_1} \sum_{k=v+1}^{n} \int_{-\infty}^{\infty} \\ \{e_k \exp[e_k s(x - y)] - \bar{e}_k \exp[\bar{e}_k s(x - y)]\} u(y) dy ds\end{aligned} \tag{3.3}$$

for $u \in L_2(-\infty, \infty)$, $\Delta \subseteqq [0, \infty)$. Suppose that $A$, $B$ are real multiplication operators $Au = f_2(x)u$, $Bu = f_1(x)u$, where

$$f_1, f_2 \in L_2(-\infty, \infty), f_1, f_2 \in C(-\infty, \infty) .$$

Then the differential operator $L^1 = L^0 + (f_1(x)f_2(x))$ determines a self-adjoint operator $H^1 = H^0 + B \cdot A$ on $L_2(-\infty, \infty)$. To show $H^1$ is almost $H^0$ smooth, but not $H^0$ smooth we must show

$$\int_0^\infty \int_{-\infty}^\infty |f_i(x)(R^0(1 \pm i\delta)E^0(\Delta_m)u)(x)|^2 dx d1 \leqq P(\Delta_m)||u|| , \qquad i = 1, 2 \tag{3.4}$$

where $\Delta_m$ is any interval $\Delta_m = (1/m, \infty)$. Since

$$R^0_m(\lambda) = R^0(\lambda)E^0(\Delta_m) = \int_{1/m}^\infty (1_1 - \lambda)^{-1} dE^0(1_1)$$

is regular for $\lambda$ real $0 \leqq \lambda \leqq 1/2m$ equation (3.4) follows if

$$\int_{1/2m}^\infty \int_{-\infty}^\infty \left| f_i(x) \int_{-\infty}^\infty G^0(x, y\colon 1 \pm i\delta)u(y)dy \right|^2 dx d1 \leqq P_1(\Delta_m)||u|| , \qquad i = 1, 2 \tag{3.5}$$

for all $m \geqq 1$. Employing (3.2) it turns out (3.5) holds with

$$P_1(\Delta_m) = (2m)^2 n |\operatorname{Re} e_{v+2}|^{-1} \pi \max_{i=1,2} \left( \int_{-\infty}^\infty f_i^2 dx \right)$$

and (3.4) with

$$P(\Delta_m) = P_1(\Delta_m) + \max_{i=1,2} \left( \int_{-\infty}^\infty f_i^2 dx \right) .$$

Now assume that the functions $f_1, f_2$ satisfy the following conditions:

(a) $f_i(x), g_i(m : x) \in C(-\infty, \infty) \cap L_1(-\infty, \infty) \cap L_2(-\infty, \infty)$ and

$$\int_{-\infty}^\infty |f_1(x)|^2 dx \left( \int_{-\infty}^\infty |f_{n+1}(x)| dx \right)^2 < 1$$

where

$$f_{i+1} = \int_x^\infty f_i dx,\ g_{i+1} = \int_x^\infty g_i dx\ , \qquad i = 3, \cdots, n\ ,$$

$$g_2(m:x) = E^0(\Delta_m)f_2 u,\ f_3 = \left(\int_x^\infty f_2^2 dx\right)^{\frac{1}{2}},\ g_3 = \left(\int_x^\infty g_2^2 dx\right)^{\frac{1}{2}},$$

*for any* $u \in C^\infty(-\infty, \infty)$, $\| u \| < 1$,

(b) $\int_{-\infty}^\infty \int_{-\infty}^\infty | f_1(x) f_2(y) G^0(x, y: \lambda_0)|^2 dxdy < \infty$

for some $\lambda_0$, $\mathrm{Im}(\lambda_0) \neq 0$. Assumption (i) is valid if there exists $N$ such that

$$\int_{-\infty}^\infty \left| f_1(x) \int_{-\infty}^\infty G^0(x, y: \lambda) g_2(m:y) dy \right|^2 dx \leqq K < 1 \tag{3.6}$$

for $m \geqq N$ and all $\lambda$ not real positive or zero. Integrating by parts $(n-1)$-times (3.6) is equivalent to

$$\int_{-\infty}^\infty \left| f_1(x) \int_{-\infty}^\infty \left(\frac{\partial^{n-1} G^0}{\partial y^{n-1}}\right) g_{n+1}(m:y) dy \right|^2 dx \leqq K < 1\ . \tag{3.7}$$

Inspection of (3.2) shows $|\partial^{n-1} G^0 / \partial y^{n-1}| \leqq 1$ for all $x, y$ and $\lambda$ not positive real. Therefore (3.7) holds if

$$\int_{-\infty}^\infty | f_1(x)|^2 dx \left(\int_{-\infty}^\infty | g_{n+1}(m:y)| dy\right)^2 \leqq K < 1\ . \tag{3.8}$$

Since $E^0(\Delta_m) f_2 u = g_2(m:x)$ converges to $f_2(x)u$, $n \to \infty$, in the $L_2(-\infty, \infty)$ norm it follows, using $(a)$, that there exists $N$ such that (3.8) holds for $m \geqq N$. The assumption (b) implies that the operator $|V|^{1/2} R^0(\lambda_0)$ has a kernel in the Schmidt class. Therefore when (a), (b) hold the wave operators exist corresponding to $L^0$, $L^1$, as a consequence of Theorem 3.

Similar results to those stated above apply to the differential operator $L^0 = -(d/dx)^2$ defined on $[0, \infty)$ with the boundary condition $u'(0) = 0$ imposed at $x = 0$. In this case we assume that $f_1, f_2$ are such that

$$f_1, f_2, f_3, g_2, g_3$$

are in $C(0, \infty) \cap L_1(0, \infty) \cap L_2(0, \infty)$ and

$$\int_0^\infty f_1^2 dx \left(\int_0^\infty | f_3 | \, dx\right)^2 < 1\ .$$

Again $L^1 = L^0 + (f_2(x) \cdot f_1(x))$ is almost smooth but not smooth with respect to $L^0$ [3, p. 381].

## References


1. T. Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin, 1966.
2. ———, *Wave operators and similarity for some non-self-adjoint operators*, Math. Ann. **169** (1966), 258-279.
3. J. Moser, *Störungstheorie des kontinuierlichen Spektrums für gewöhnliche Differentialgleichungen zweiter Ordnung*, Math. Ann. **125** (1953), 366-393.


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