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GENERALIZED BOL FUNCTIONAL EQUATION

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Each identity in a group or in a quasigroup induces a generalized identity (functional equation) in a class of quasigroups. Generalized associativity, generalized bisymmetry and generalized distributivity are examples of such generalized identities. From the left Bol identity

$$x(y(xz)) = (x(yx))z$$

on a quasigroup, we obtain a generalized Bol identity on a class of quasigroups:

$$A_1(x, A_2(y, A_3(x, z))) = A_4(A_5(x, A_6(y, x)), z)\ ,$$

where the A_i's are quasigroup operations on a set Q. The general solution of this generalized Bol functional equation is obtained by reducing it to another functional equation

$$P(x, y + S(x, z)) = P(x, y + \alpha(x)) + z$$

where P and S are quasigroup operations on Q and $\alpha(x) = S(x, 0)$. If the operations in the last functional equation are considered on real numbers (or groups), then the solution of this equation is obtained.

One of the most important identities considered in the theory of quasigroups is Bol identity. A loop $Q(\cdot)$ isc alled a left Bol loop [2] if the following identity

$$x(y(xz)) = (x(yx))z\ , \tag{1}$$

holds for every $x, y, z \in Q$. The identity (1) is called the left Bol identity. The right Bol identity is defined analogously

$$((zx)y)x = z((xy)x)\ . \tag{2}$$

For more information of algebraic properties of Bol loops, see for example [4]. If a loop is both a right and a left Bol, then it is a Moufang loop, i.e. one of the following Moufang identities are satisfied:

$$x(y(xz)) = ((xy)x)z\ , \tag{3}$$

$$((zx)y)x = z(x(yx))\ . \tag{4}$$

It is easily seen that (3) is a particular case of (2); if $Q(\cdot)$ satisfies the elasticity law $(xy)x = x(yx)$, then (1) implies (3). On the other hand the left Moufang identity (3) does not imply (1), see for example [3].

Each identity in an universal algebra defines a generalized identity which is obtained from the given identity by replacing operations of the same arity (number of variables) by different operations of the same arity. Generalized associativity $A_1(A_2(x, y), z) = A_3(x, A_4(y, z))$, generalized bisymmetry $A_1(A_2(x, y), A_3(u, v)) = A_4(A_5(x, u), A_6(y, v))$, generalized distributivity $A_1(x, A_2(y, z)) = A_3(A_4(x, y), A_5(x, z))$ are examples of such generalized identities. These identities considered as functional equations were studied by many authors, for references see [1].

To the left Bol identity (1) corresponds the generalized Bol (left identity):

$$A_1(x, A_2(y, A_3(x, z))) = A_4(A_5(x, A_6(y, x)), z) . \tag{5}$$

The corresponding identity for the right Bol identity is

$$B_1(B_2(B_3(z, x), y), x) = B_4(z, B_5(B_6(x, y), x)) . \tag{6}$$

Of course all operations A_i, $B_j\,(i, j = 1, 2, \cdots, 6)$ in (5) and (6) are defined on the same set Q.

We shall consider the equations (5) and (6) on quasigroups, that is, we assume that all A_i and B_j are quasigroups (quasigroup operations). In the next sections, we reduce the equation (s) to a simpler one containing two quasigroups and one loop, and we give a full solution of this equation under some suppositions. For the definitions and results on quasigroups and loops, see for example [2], [3].

2. We shall use the following notations. Let A be a binary operation defined on the set Q. We denote the translations of A by

$$L_A(a)x = A(a, x) , \qquad R_A(a, x) = A(x, a) . \tag{7}$$

If A is one of the operations A_i $(i = 1, 2, \cdots, 6)$ from (5) then we shall write $L_i(a)$ instead of $L_{A_i}(a)$ and moreover, if a is a fixed element k of Q, then we shall write L_i instead of $L_i(k)$. Similar notations are used for right translations.

Let 0 be a fixed element of the set Q. We denote by $L_A(0) = L$, $R_A(0) = R$ and

$$x \underset{A}{+} y = A(R^{-1}x, L^{-1}y) . \tag{8}$$

Then $Q(\underset{A}{+})$ is a loop [3] with the neutral element $A(0, 0) = 0_A$.

Let all the operations in (5) be quasigroup oparations. Then L_i's and R_i's are permutations of Q. If $x = k$ in (5), then from (7), we have

$$L_1A_2(y, L_3z) = A_4(L_5R_6y, z) ,$$

that is,

$$A_4(y, z) = L_1 A_2(R_6^{-1} L_5^{-1} y, L_3 z) . \tag{9}$$

Using (5) and (9), we have

$$A_1(x, A_2(y, A_3(x, z))) = L_1 A_2(R_6^{-1} L_5^{-1} A_5(x, A_6(y, x), L_3 z). \tag{10}$$

With $z = k$ in (10), we get using (7),

$$A_1(x, A_2(y, R_3 x)) = L_1 R_2' R_6^{-1} L_5^{-1} A_5(x, A_6(y, x)) , \tag{11}$$

where $R_2' = R_2(L_3 k)$. From (10) and (11), we obtain

$$L_1^{-1} A_1(x, A_2(y, A_3(x, z))) = A_2(R_2'^{-1} L_1^{-1} A_1(x, A_2(y, R_3 x)), L_3 z) . \tag{12}$$

Let

$$C_1(x, y) = L_1^{-1} A_1(x, y) . \tag{13}$$

Now (12) and (13) yield,

$$C_1(x, A_2(y, A_3(x, z))) = A_2(R_2'^{-1} C_1(x, A_2(y, R_3 x)), L_3 z) ,$$

that is,

$$C_1(x, A_2(R_2'^{-1} y, R_3 R_3^{-1} A_3(x, z))) = A_2(R_2'^{-1} C_1(x, A_2(R_2'^{-1} y, R_3 x)), L_3 z) . \tag{14}$$

Let

$$C_2(x, y) = A_2(R_2'^{-1} x, R_3 y) , \qquad C_3(x, y) = R_3^{-1} A_3(x, L_3^{-1} R_3 y) . \tag{15}$$

With the help of (15), (14) can be rewritten as,

$$C_1(x, C_2(y, C_3(x, R_3^{-1} L_3 z))) = C_2(C_1(x, C_2(y, x)), R_3^{-1} L_3 z) ,$$

that is,

$$C_1(x, C_2(y, C_3(x, z))) = C_2(C_1(x, C_2(y, x)), z) . \tag{16}$$

From (13) and (15), it follows that C_1, C_2 and C_3 are quasigroup operations on Q, since L_1, L_3, R_3 and R_2' are permutations of Q.

As every quasigroup is isotopic to a loop [3], we can assume that C_2 is isotopic to a loop, that is, C_2 satisfies

$$C_2(x, y) = Rx + Ly , \quad \text{where } R \text{ and } L \text{ are as in (8)} . \tag{17}$$

Then $Q(+)$ is a loop. By (17), (16) becomes,

$$C_1(x, Ry + LC_3(x, z)) = RC_1(x, Ry + Lx) + Lz ,$$

that is,

$$C_1(x, y + LC_3(x, z)) = RC_1(x, y + Lx) + Lz . \tag{18}$$

Now define

$$P(x, y) = C_1(x, y), \qquad S(x, y) = LC_3(x, L^{-1}y) . \tag{19}$$

Evidently P and S are quasigroup operations on Q. Using (19), we obtain from (18),

$$P(x, y + S(x, Lz)) = RP(x, y + Lx) + Lz ,$$

that is,

$$P(x, y + S(x, z)) = RP(x, y + Lx) + z . \tag{20}$$

Putting $z = 0$ in (20), we have

$$P(x, y + S(x, 0)) = RP(x, y + Lx) ,$$

and thus, we get

$$P(x, y + S(x, z)) = P(x, y + \alpha(x)) + z , \tag{21}$$

where $\alpha(x) = S(x, 0)$.

Hence from (9), (13), (15), (17) and (19), results

$$\left\{\begin{aligned} A_1(x, y) &= L_1C_1(x, y) = L_1P(x, y) \\ A_2(x, y) &= C_2(R_2'x, R_3^{-1}y) = RR_2'x + LR_3^{-1}y \\ A_3(x, y) &= R_3C_3(x, R_3^{-1}L_3y) = R_3L^{-1}S(x, LR_3^{-1}L_3y) \\ A_4(x, y) &= L_1A_2(R_6^{-1}L_5^{-1}x, L_3y) = L_1(RR_2'L_5^{-1}x + LR_3^{-1}L_3y) , \end{aligned}\right. \tag{22}$$

where P and S satisfy (21).

With $L_1 = \phi$, $RR_2' = \lambda$, $LR_3^{-1} = \mu$, $L_3 = \psi$, $R_2'R_6^{-1}L_5^{-1} = \theta$, (22) can be rewritten as,

$$\left\{\begin{aligned} A_1(x, y) &= \phi P(x, y) \\ A_2(x, y) &= \lambda x + \mu y \\ A_3(x, y) &= \mu^{-1}S(x, \mu\psi y) \\ A_4(x, y) &= \phi(R\theta x + \mu\psi y) , \end{aligned}\right. \tag{23}$$

where P and S satisfy (21) and $\phi, \lambda, \mu, \psi, R$ and θ are permutations on Q.

From (11) and (23), we obtain

$$\begin{aligned} A_1(x, A_2(y, R_3x)) &= \phi P(x, \lambda y + \mu R_3x) \\ &= \phi P(x, \lambda y + Lx) \\ \text{also} \quad &= \phi\theta A_5(x, A_6(y, x)) , \end{aligned}$$

thus

$$A_5(x, A_6(y, x)) = \theta^{-1}P(x, \lambda y + Lx) . \tag{24}$$

Thus we have proved a part of the following,

THEOREM 1. *Let Q be an arbitrary set. Let A_i $(i = 1, \cdots 6)$ be quasigroup operations satisfying* (5). *Then all the solutions of the functional equation* (5) *are given by* (23) *and* (24) *where $\phi, \lambda, \mu, \psi, R, \alpha, \theta$ and L are arbitrary permutations of Q, the loop operation $+$ and the quasigroup operations P and S satisfy* (21). *Conversely the A_i's $(i = 1, 2, \cdots, 6)$ given by* (23) *and* (24), *where P and S satisfy* (21), *satisfy the generalized Bol equation* (5).

By a straight forward computation, it is easy to verify the converse part.

REMARK 1. The solutions of the right Bol functional equation (6), can be obtained from (5), by replacing all the B_i's in (6) by the D_i's where

$$D_i(x, y) = B_i(y, x) .$$

REMARK 2. The generalized Moufang functional equation

$$A_1(x, A_2(y, A_3(x, z))) = A_4(A_5(A_6(x, y), x), z)$$

can also be reduced to (21). In that by the same computation, we obtain (12), from which (16) and finally (21). All the solutions are similar to (23). Only difference is (24), where instead of (24), we get

$$A_5(A_6(x, y), x) = \theta^{-1}P(x, \lambda y + Lx) .$$

3. As we have seen in § 2, the solution of the Bol functional equation (5) is reduced to that of (21). Let us now consider this equation (5) on the set of real numbers $\mathbf{R}$; and let us suppose that $Q(+)$ is the additive group of real numbers. So, we have to consider (21) on $\mathbf{R}$.

Letting $S(x, z) = t$ in (21), we get, using S as a quasigroup operation

$$P(x, y + t) = P(x, y + \alpha(x)) + S^{-1}(x, t), \quad x, y, t \in \mathbf{R} ,$$

where $\alpha(x) = S(x, 0)$. Thus, we obtain

$$\lambda_x(y + t) = \mu_x(y) + \nu_x(t), \qquad \text{for all } y, t \in \mathbf{R} , \tag{25}$$

where

$$\lambda_x(u) = P(x, u), \ \mu_x(u) = P(x, u + \alpha(x)), \ \nu_x(u) = S^{-1}(x, u) . \tag{26}$$

Equation (25) is the well known Pexider equation. Hence there exists an additive function A_x on $\mathbf{R}$ satisfying

$$A_x(u+v) = A_x(u) + A_x(v) \tag{27}$$

for all $u, v \in \mathbf{R}$, such that

$$\begin{cases} \lambda_x(u) = C(x) + A_x(u) , \\ \mu_x(u) = b(x) + A_x(u) , \\ \nu_x(u) = d(x) + A_x(u) , \end{cases} \tag{28}$$

where $b(x)$, $c(x)$, $d(x)$ are constants depending on x with $c(x) = b(x) + d(x)$. With the notation

$$F(x, u) = A_x(u) , \tag{29}$$

we obtain from (26), (28) and (29),

$$P(x, u) = c(x) + F(x, u) , \tag{30}$$

$$S^{-1}(x, u) = d(x) + F(x, u) , \tag{31}$$

where F is additive in the second variable for each fixed x. From (30), we see that F is a right quasigroup, that is,

(32) $F(a, x) = b$, has a unique solution for all a, b.

If in (31), we put $S^{-1}(x, u) = w$, then we have

$$\begin{aligned} d(x) + F(x, u) &= w , \\ S(x, w) &= u . \end{aligned}$$

Thus,

$$S(x, d(x) + F(x, u)) = u . \tag{33}$$

Since $S(x, 0) = \alpha(x)$, we have $S^{-1}(x, \alpha(x)) = 0$. Thus, from (31) with $u = \alpha(x)$, we get $d(x) = F(x, -\alpha(x))$, using F additive in the second variable. Hence (33) becomes

$$S(x, F(x, u - \alpha(x))) = u ,$$

that is

$$S(x, F(x, y)) = y + \alpha(x) ,$$

from which follows using (32),

$$S(x, y) = \alpha(x) + F^{-1}(x, y) \tag{34}$$

Therefore, we have proved the following:

THEOREM 2. *Let* $Q(+)$ *be the additive group of real numbers. Then the general solution of* (21) *is given by* (30) *and* (34) *where*

F is an arbitrary right quasigroup which is additive in the second variable and $c(x)$ and $\alpha(x)$ are arbitrary functions. Conversely if P and S are given by (30) *and* (34) *with F additive in the second variable, then* (21) *holds.*

The converse part can be obtained by easy computation.

REMARK 3. In order for P and S to be quasigroups, we need the following conditions on $C(x)$ and $\alpha(x)$; $C(x) + F(x, a) = b$ and $\alpha(x) + F^{-1}(x, a) = b$ should have unique solutions for given a and b. But if we require only P and S to be right quasigroups, then we do not need these conditions and the solution of (21) is given by (30) and (34) for arbitrary $C(x)$ and $\alpha(x)$.

REMARK 4. If we take P to be monotonic in the second variable, then from (27), (29) and (30), we see that $A_x(u)$ is continuous and, for $A_x \not\equiv 0$, $A_x(u) = l(x)u$, for arbitrary $l(x)$. Hence $P(x, u) = C(x) + l(x)u$ and $S(x, u) = \alpha(x) + u/l(x)$.

REMARK 5. Instead of the additive group of real numbers, we can take an arbitrary group and consider the Pexider type equation on this group. The general solution of (21) is given by (28) and hence by (30) and (34). But the constant functions $c(x)$, $b(x)$ and $d(x)$ in (28) should be written in a proper way [5].

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