PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 3, 1970

ON SOME TYPES OF COMPLETENESS IN TOPOLOGICAL VECTOR SPACES

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A type of closed family is defined for which problems of existence, construction and approximation are examined in some topological vector spaces or their conjugate spaces.

1. Definition of a type of completeness.

1.1. Let E denote a real or complex topological vector space, E' its conjugate space, I a set of indices (occasionally $I = N$), $(x_i)_{i\in I}$ a family of elements in E, V the vector space of families $\lambda = (\lambda_i)_{i\in I}$ of real or complex numbers, where $\lambda_i = 0$ except for a finite number of indices, F a semi-norm on V. It is recalled that the family $(x_i)_{i\in I}$ is *total* if the subspace generated by the x_i is everywhere dense in E, and it is *closed* (or complete) if $\varphi = 0$ is the only $\varphi \in E'$ satisfying $\varphi(x_i) = 0$ for every $i \in I$. If E is a locally convex space, these definitions are equivalent; then, the existence of a closed sequence is for E a necessary and sufficient condition of separability. New types of completeness have been studied for $I = N$ by Ph. Davis and Ky Fan [5] in a normed space, and by S. Ja Havinson [6] in a locally convex topological space. The following definitions seem to be more convenient than Havinson's point of view for the generalization to locally convex spaces of some results given by [5].

DEFINITION 1.2. According to precedent notations, we said that *the family* $(x_i)_{i\in I}$ is *F-closed in* E if:

(C) *the conditions* $\varphi \in E'$; $|\sum_{i\in I}\lambda_i\varphi(x_i)| \leqq F(\lambda)$ *for all* $\lambda = (\lambda_i) \in V$ *imply* $\varphi = 0$.

Clearly a F-closed family is a closed family. Let (a_n) be a given sequence of nonnegative numbers, let $p \geqq 1$, $1/p + 1/q = 1$. Writing $F_{(a_n)}(\lambda) = \sum_{n\in N} |\lambda_n| a_n$; $F_p(\lambda) = (\sum_{n\in N} |\lambda_n|^q)^{1/q}$ if $p > 1$; $F_p(\lambda) = \sup_{n\in N} |\lambda_n|$ if $p = 1$, for each $\lambda = (\lambda_n) \in V$, we see that definitions of [5] are special cases of the Definition 1.2.

2. Characteristic property for F-completeness.

THEOREM 2.1. *Let E be a locally convex space. Let $\mathscr{P} = (p_j)_{j\in J}$ be a basis of continuous seminorms on E defining the topology of E. The family $(x_i)_{i\in I}$ of elements in E is F-closed if and only if:*

(P) *For any* $x \in E$, *for any* $p_j \in \mathscr{P}$, *and for any* $\varepsilon > 0$, *there exists* $\lambda = (\lambda_i)_{i \in I} \in V$ *such that*:

$$(1) \qquad p_j\Big(x - \sum_{i \in I} \lambda_i x_i\Big) < \varepsilon \; ; \qquad F(\lambda) < \varepsilon \; .$$

Proof. "*If part*". Let $\varphi \in E'$ be such that $|\sum_{i \in I} \lambda_i \varphi(x_i)| \leqq F(\lambda)$ for all $\lambda = (\lambda_i) \in V$.

There exist a constant $K > 0$ and a seminorm p_k such that

$$|\varphi(y)| \leqq K p_k(y) \quad \text{for all} \quad y \in E \; .$$

Given any $x \in E$, there exists $\mu = (\mu_i) \in V$ such that

$$p_k\Big(x - \sum_{i \in I} \mu_i x_i\Big) < \varepsilon \; ; \qquad f(\mu) < \varepsilon$$

We write:

$$\varphi(x) = \varphi\Big(x - \sum_{i \in I} \mu_i x_i\Big) + \sum_{i \in I} \mu_i \varphi(x_i) \; ,$$

then:

$$|\varphi(x)| \leqq (1 + K)\varepsilon \; .$$

This shows that $\varphi(x) = 0$ for all $x \in E$. Hence $\varphi = 0$.

"*Only if part*". Let us put on V the topology defined by the seminorm F, and consider the topological product $E \times V$ with its locally convex product topology. Let us denote by $u_i = (\delta_{ik})_{k \in I}$ the family whose i-th term is 1 and all other terms are 0; by ξ_i the element of $E \times V$ defined by $\xi_i = (x_i, u_i)$; and by $\mathscr{F}$ the subspace of $E \times V$ spanned by the ξ_i. The condition (P) of theorem is implied by the following:

(Q) $(\forall \varphi \in E')(\forall \psi \in V')((\forall i \in I)(\varphi(x_i) + \psi(u_i) = 0) \Rightarrow \varphi = 0)$ because if (P) is not verified we have:

$(\bar{P})$ $(\exists x_0 \in E)(\exists p_j \in \mathscr{F})(\exists \varepsilon > 0)(\forall \lambda = (\lambda_i) \in V)((\bar{1}) p_j(x_0 - \sum_{i \in I} \lambda_i x_i) \geqq \varepsilon$ or $F(\lambda) \geqq \varepsilon)$. The existence of x_0 implies that $x_0 \neq 0$ otherwise the two inequalities of $(\bar{1})$ are contradicted by the choice of $\lambda = 0$. Let us denote by B_i the "open ball" whose centre is $(x_0, 0)$ which is the product of open balls $b_i = \{y \in F / p_j(x_0 - y) < \varepsilon\}$ and

$$\beta = \{v \in V / F(v) < \varepsilon\} \; .$$

It is easily seen that there is no element of $\mathscr{F}$ in B_i, the elements of $\mathscr{F}$ being:

$$\sum_{i \in I} \lambda_i \xi_i = \Big(\sum_{i \in I} \lambda_i x_i, \sum_{i \in I} \lambda_i u_i\Big) \quad \text{where} \quad \lambda = (\lambda_i) \in V$$

because otherwise we should have:

$$p_j\Big(x_0 - \sum_{i\in I}\lambda_i x_i\Big) < \varepsilon \;;\; F\Big(\sum_{i\in I}\lambda_i u_i\Big) = F(\lambda) < \varepsilon$$

and that is in contradiction with the two inequalities of $(\bar{1})$.

Then $(x_0, 0)$ is the centre of an open "ball" which contains no element of $\mathscr{F}$ and $(x_0, 0) \in E \times V - \bar{\mathscr{F}}$. And then there exists a linear continuous functional Φ on $E \times V$ such that $\Phi(x_0, 0) \neq 0$ and vanishing on $\bar{\mathscr{F}}$ and $\mathscr{F}$ (cf., e.g., [8, Th. 3.8. E]).

Now Φ is written in unique way: (cf., e.g., [4, Ch. IV])

$$\Phi\colon (x, v) \longrightarrow \varphi(x) + \psi(v) \quad \text{where} \quad \varphi \in E' \,,\; \psi \in V' \,,$$

and we have:

$$\varphi(x_0) = \Phi(x_0, 0) \neq 0 \;.$$

Then $\varphi \neq 0$, and that involves:

$$(\bar{Q})(\exists\varphi \in E')(\exists\psi \in V')((\forall i \in I)(\varphi(x_i) + \psi(u_i) = 0) \text{ and } \varphi \neq 0) \;.$$

The implication $(\bar{P}) \Rightarrow (\bar{Q})$ is proved, i.e., the implication $(Q) \Rightarrow (P)$.

It only remains to prove that condition (C) implies (Q). For a given $\varphi \in E'$, if there exists $\psi \in V'$ such that for all $i \in I$

$$\varphi(x_i) + \psi(u_i) = 0 \;;$$

we have, for all $\lambda = (\lambda_i) \in V$:

$$\Big|\sum_{i\in I}\lambda_i\varphi(x_i)\Big| = \Big|\sum_{i\in I}\lambda_i\psi(u_i)\Big| = \Big|\psi\Big(\sum_{i\in I}\lambda_i u_i\Big)\Big| = |\,\psi(\lambda)\,| \leqq \|\,\psi\,\|\, F(\lambda)$$

where $\|\;\|$ is the usual norm on V'. Hence $\varphi = 0$.

REMARKS 2.2. It is easily seen that for any semi norm F on V a F-closed family is total because the first of inequalities (1) of Theorem 2.1 proves that the subspace generated by (x_i) is everywhere dense in E. Then taking $F = 0$ we have: In a locally convex space a closed family is total. The converse is easily proved and we find again the equivalence of these two properties. If E is a normed space, the Theorem 2.1 has as corollaries Theorems 1 and 2 of [5]; moreover it extends them in the case of families in locally convex spaces. At last it results of the demonstration that the condition (Q) is also necessary and sufficient for the family (x_i) to be F-closed. We shall make use of this result at § 6.

3. F-closed families in the conjugate of a locally convex Hausdorff space.

PROPOSITION 3.1. *Let E be a locally convex Hausdorff semi-reflexive space. A family $(L_i)_{i\in I}$ of elements in E' is F-closed for the strong topology of E' if and only if the conditions:*

$$x \in E;\ |\textstyle\sum_{i\in I} \lambda_i L_i(x)| \leqq F(\lambda) \text{ for all } \lambda = (\lambda_i) \in V$$

imply $x = 0$.

We do not give the proof which results from the fact that E being semi-reflexive, and $\varphi \in E''$ (second conjugate of E) is written

$$\varphi\colon L \longrightarrow L(x) \quad \text{where} \quad x \in E\,,$$

for each $L \in E'$, and that, by Hahn-Banach, $x = 0$ is the only $x \in E$ for which $L(x) = 0$ for all $L \in E''$.

EXAMPLES 3.2. We denote by $\mathscr{H}(\Omega)$ the space of holomorphic functions is the unit disc, with the topology of compact convergence. Let $(z_n)_{n\in N}$ be a sequence of complex numbers such that $0 < |z_n| < 1$, $\lim_{n\to\infty} z_n = 0$. We denote by $(L_n)_{n\in N}$ the sequence of elements in $[\mathscr{H}(\Omega)]'$ defined by $L_n(f) = f(z_n)$ for all $n \in N$ and all $f \in \mathscr{H}(\Omega)$. Let F be a semi norm on V:

3.2.1. If $\lim_{n\to\infty} (F(u_n)/|z_n^k|) = 0$ for all $k \in N$ where $u_n = (\delta_{nm})_{m=0}^{\infty}$, then $(L_n)_{n\in N}$ is F-closed for the strong topology of $[\mathscr{H}(\Omega)]'$.

3.2.2. If $F = F_p$ with p finite (c.f. § 1). Then $(L_n)_{n\in N}$ is F_p-closed for the strong topology of $[\mathscr{H}(\Omega)]'$ if and only if the series $\sum_0^\infty |z_n|^\alpha$ diverges for all $\alpha > 0$.

4. Construction of F-closed sequences in Fréchet spaces.

PROPOSITION 4.1. *Let $(x_n)_{n\in N}$ be a closed sequence of elements in a Fréchet space $\mathscr{F}$, whose topology is defined by an enumerable family of continuous seminorms $(p_i)_{i\in N^+}$ such that*

$$(1) \qquad \sum_1^\infty 2^{-i} p_i(x_n) = A_n < +\infty\ ;\ \overline{\lim}\, A_n^{1/n} \leqq 1\,.$$

Let $(L_n)_{n\in N}$ be a sequence of elements in $[\mathscr{H}(\Omega)]'$ F-closed for the strong topology. Then the sequence $(y_n)_{n\in N}$ of elements in $\mathscr{F}$ associated with (L_n) by

$$(2) \qquad y_n = \sum_0^\infty L_n(\varphi_k)x_k \qquad (\varphi_k : z \longrightarrow z^k)$$

for all $n \in N$ *is F-closed.*

Proof. From the continuity of $L_n \in [\mathscr{H}(\Omega)]'$ we have

$$\overline{\lim_{k\to\infty}} \, |L_n(\varphi_k)|^{1/k} < 1 .$$

The topology of $\mathscr{F}$ being equivalent to the metric defined by the distance:

$$d(x, y) = \sum_{i=1}^\infty 2^{-i} \frac{p_i(x-y)}{1 + p_i(x-y)} ,$$

we have, with $p(x) = d(x, 0)$:

$$\begin{aligned} p[L_n(\varphi_k)x_k] &= \sum_{i=1}^\infty 2^{-i} \frac{p_i[L_n(\varphi_k)x_k]}{1 + p_i[L_n(\varphi_k)x_k]} \leqq [L_n(\varphi_k) \left| \sum_{i=1}^\infty 2^{-i} p_i(x_k) \right| \\ &= |L_n(\varphi_k)| \, A_k . \end{aligned}$$

Then from (1) we have:

$$\overline{\lim} \, [p[L_n(\varphi_k)x_k]]^{1/k} \leqq \overline{\lim} \, |L_n(\varphi_k)|^{1/k} \, \overline{\lim} \, A_k^{1/k} < 1 .$$

That proves the convergence of the numerical series $\sum_0^\infty p[L_n(\varphi_k)x_k]$ and as $\mathscr{F}$ is complete, the convergence of the series giving y_n. Now, let $\varphi \in \mathscr{F}'$ be such that, for all $\lambda = (\lambda_n) \in V$:

$$\left| \sum_{n \in N} \lambda_n \varphi(y_n) \right| \leqq F(\lambda) .$$

The power series $\sum_0^\infty \varphi(x_k)z^k$ has a radius of convergence $R \geqq 1$. Indeed there exists a constant C, and an index i such that $|\varphi(x)| \leqq Cp_i(x)$ for all $x \in \mathscr{F}$, and from (1) we have:

$$\overline{\lim_{n\to\infty}} \, [p_i(x_n)]^{1/n} \leqq 1$$

for each $i \in N^*$. Finally

$$\overline{\lim} \, |\varphi(x_k)|^{1/k} \leqq \lim C^{1/k} \, \overline{\lim} \, [p_i(x_k)]^{1/k} \leqq 1 .$$

Then the indicated series defines a function holomorphic at least in Ω,

$$f = \sum_0^\infty \varphi(x_k)\varphi_k \in \mathscr{H}(\Omega) \quad \text{and} \quad L_n(f) = \sum_0^\infty \varphi(x_k)L_n(\varphi_k) = \varphi(y_n) .$$

Then we have:

$$\left|\sum_0^\infty \lambda_n L_n(f)\right| \leqq F(\lambda) \quad \text{for all} \quad \lambda = (\lambda_n) \in V\ .$$

As (L_n) is F-closed in $[\mathscr{H}(\Omega)]'$ that inequality implies $f = 0$, from which we gather $\varphi(x_k) = 0$ for all k and as (x_n) is closed in $\mathscr{F}$: $\varphi = 0$.

EXAMPLES 4.5.

4.5.1. If $\mathscr{F}$ is a Banach space for which $p_i(x) = \|x\|$ for every i, the precedent condition (1) of 4-1 is written $\overline{\lim}\,\|x_n\|^{1/n} \leqq 1$, and we find again, as a corollary, the Theorem 3 of [5].

4.5.2. Let $(a_n)_{n \in N}$ be a given sequence of numbers $\geqq 0$, $(z_n)_{n \in N}$ a sequence of complex numbers such that, K being a constant $\geqq 0$; $a_n^{1/n} \leqq K|z_n|$; $0 < |z_n| < 1$; $\lim z_n = 0$. Let $g = \sum_0^\infty \alpha_k \varphi_k \in \mathscr{H}(\Omega)$ with $\alpha_k \neq 0$ for all k. Then, the sequence (y_n) of elements in $\mathscr{H}(\Omega)$ defined by $y_n(z) = g(zz_n)$ for all $z \in \Omega$ is $F_{(a_n)}$-closed.

5. Existence of semi norms associated to a given family. For a given family $(x_i)_{i \in I}$ of elements in E when does one know whether there exists a semi norm F such that (x_i) is F-closed?

DEFINITION 5.1. On the space V we defined the order relation (not total) $\leqq$ by: $\lambda' \leqq \lambda$ if every term $\neq 0$ in λ' is a term in λ with the same index. We shall denote F^* any semi norm increasing on V.

THEOREM 5.2. *Let us suppose $(x_i)_{i \in I}$ to be topologically free* ([4], Ch. I]). *If $(x_i)_{i \in I}$ is F^*-closed, then $F^* = 0$.*

Proof. Let $u_i = (\delta_{jk})_{k \in I} \in V$. If $F^* = 0$ there exists $j \in I$ such that $F^*(u_j) \neq 0$. Let E_j be the subspace spanned by the elements of (x_i) other than x_j. Then by Hahn-Banach, there exists $\varphi \in E'$ such that $\varphi(\overline{E}_j) = 0$ and $\varphi(x_j) = 1$. Let us write $\psi = F^*(u_j)\varphi$: ψ does not vanish, and that is in contradiction, if (x_i) is F^*-closed, with

$$\left|\sum_{i \in I} \lambda_i \psi(x_i)\right| = |\lambda_j|\, F^*(u_j) = F^*(\lambda_j u_j) \leqq F^*\Big(\sum_{i \in I} \lambda_i u_i\Big) = F^*(\lambda)$$

for all $\lambda = (\lambda_i) \in V$.

COROLLARY 5.3. *In a topological vector space* (*resp. metrizable and complete*) *any Schauder basis* (*resp. any basis*) *is F^*-closed if and only if $F^* = 0$.*

COROLLARY 5.4. *In the conjugate of a locally convex Hausdorff* *semi reflexive space (resp. Eréchet semi reflexive space) the sequence of the coefficient functionals associated with any Schauder basis (resp. any basis) is strongly F^*-closed if and only if $F^* = 0$.* (See, e.g., [2, Ch. VII, § 3], [7, p. 431–432], [1, Th. 2]).

EXAMPLE 5.5. The sequence $(\varphi_n)_{n\in N}\varphi_n\colon z \to z^n$ is a Schauder basis for $\mathscr{H}(\Omega)$. Then it is F^*-closed (e.g., $F^* = F_{(a_n)}$ or $F^* = F_p$) if and only if $F^* = 0$. From 5.4 we have the same results with the sequence of the coefficient functionals in $[\mathscr{H}(\Omega)]'$: $(L_n)_{n\in N}$ such that $L_n(f) = (f^{(n)}(0)/n!)$ for all $f \in \mathscr{H}(\Omega)$ and also from 5.3 because $\mathscr{H}(\Omega)$ being reflexive, (L_n) is a strong Schauder basis for $[\mathscr{H}(\Omega)]'$.

6. Neighboring F-closed families in Paley-Wiener sense. (On the Paley-Wiener theorem see, e.g., [3, Th. 1.1]; [5, Th. 4]; [1]). We shall prove that F-completeness of a family (x_i) is carried into another family (y_i) closed to (x_i) in the sense of the following:

THEOREM 6.1. *Let E be a locally convex space. Let $\mathscr{P} = (p_j)_{j\in J}$ a basis of continuous seminorms on E defining the topology of E. Let $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ be two families of elements in E. Let us suppose that there exists a family $(k_j)_{j\in J}$ of real numbers $(0 \leqq k_j < 1)$ such that, for all $\lambda \in (\lambda_i) \in V$:*

$$(1) \qquad p_j\Big(\sum_{i\in I} \lambda_i(x_i - y_i)\Big) \leqq k_j p_j\Big(\sum_{i\in I} \lambda_i x_i\Big)$$

for every $j \in J$. Then, if (x_i) is F-closed, so is (y_i).

Proof. The notations are those of Theorem 2.1. According to Remarks 2.2, to prove that (y_i) is F-closed we must show that the conditions:

$$(2) \qquad \varphi \in E' , \ \psi \in V' \quad \varphi(y_i) + \psi(U_i) = 0$$

for all $i \in I$ imply $\varphi = 0$.

Then, Let φ and ψ be such that the conditions (2) are verified, and let

$$\alpha_i = \varphi(x_i) + \psi(U_i) = \varphi(x_i - y_i) .$$

There exist a constant $K > 0$ and an index $j \in I$ such that:

$$|\varphi(x)| \leqq Kp_j(x) \quad \text{for all} \quad x \in E ,$$

and as $\|\varphi\|_j = \inf\{H/(\forall x \in E)\varphi(x) \leqq Hp_j(x)\}$ is a norm on E', we also have:

$$|\varphi(x)| \leqq \|\varphi\|_j p_j(x) \quad \text{for all} \quad x \in E .$$

Then, from (1), for all $\lambda = (\lambda_i) \in V$:

$$\left|\sum_{i \in I} \lambda_i \alpha_i\right| \leqq \|\varphi\|_j p_j\Big(\sum_{i \in I} \lambda_i(x_i - y_i)\Big) \leqq k_j \|\varphi\|_j p_j\Big(\sum_{i \in I} \lambda_i x_i\Big) .$$

By the generalization of Banach's result ([2, Ch. IV, § 3, Th. 5]), this proves the existence of $\varphi_1 \in E'_{\tau_j}$ (that is to say a linear functional on E continuous for the semi norm p_j topology and consequently, for the initial topology of E) such that:

$$\|\varphi_1\|_j \leqq k_j \|\varphi\|_j \quad \text{and} \quad \varphi_1(x_i) = \alpha_i$$

for all $i \in I$. Then we have:

$$(\varphi - \varphi_1)(x_i) + \psi(v_i) = 0 \quad \text{for all} \quad i \in I .$$

As $\varphi_1 \in E'$ and $\varphi - \varphi_1 \in E'$, according to F-completeness of (x_i) we have, from (2) $\varphi - \varphi_1 = 0$. Then $\|\varphi\|_j \leqq k_j \|\varphi\|_j$ with $0 \leqq k_i < 1$. Hence $\varphi = 0$.

COROLLARY 6.2. *If (x_i) and (y_i) satisfy the hypothesis of Theorem 6.1 with $0 \leqq k_i \leqq 1/2$ for all $i \in I$, then these families are simultaneously F-closed or not.*

COROLLARY 6.3. *Let E be a Fréchet space, of which the topology is defined by an enumerable family of continuous norms $(\|\ \|_i)_{i \in N^*}$. Let us suppose that E is a unitary normed algebra for each $\|\ \|_i$. Let $(x_n)_{n \in N}$ be an absolutely convergent basis in E, and $(\varepsilon_n)_{n \in N}$ be a sequence of elements in E:*

$$\varepsilon_n = \sum_{k=0}^{\infty} \varphi_k(\varepsilon_n) x_k ,$$

such that

$$\sum_{k=0}^{\infty} \|\varphi_k(\varepsilon_n) x_k\|_i \leqq l$$

(a given constant) for every $i \in N^*$. Let $(y_n)_{n \in N}$ be the sequence of elements in E defined by

$$y_n = x_n(1 + \varepsilon_n)$$

for every $n \in N$. Then

(i) *If $l < 1$ and if there exists a semi norm F on V such that (x_n) is F-closed, so is (y_n).*

(ii) *If $l < 1/2$, (x_n) and (y_n) are simultaneously F-closed or not. Particularly (y_n) is F^*-closed if and only if $F^* = 0$.*

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Received November 7, 1969.

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