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ON A THEOREM OF M. IZUMI AND S. IZUMI

SHIVA NARAIN LAL

This paper establishes a theorem on the absolute Nörlund summability of Fourier series which generalizes and unifies generalizations by the author and by M. and S. Izumi of an earlier result by McFadden.

Let $\sum a_n$ be a series with partial sums S_n and let p_n be a sequence of real constants with

$$P_n = \sum_{v=0}^{n} p_v\ , \quad p_0 > 0\ , \quad P_{-1} = p_{-1} = 0\ .$$

The series $\sum a_n$ is said to be summable $|N, p_n|$ if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty\ ,$$

where

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} S_v\ . \tag{1.1}$$

We write $P(t) = P_{[t]}$ and in the sequel we assume that p_n is nonnegative, nonincreasing and $\lim_{n\to\infty} p_n = 0$.

2. Let $f(t)$ be a periodic function with period 2π and integrable (L) in $(-\pi, \pi)$. The Fourier series of $f(t)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t)\ ,$$

where a_n and b_n are given by the usual Euler-Fourier formulae. We write

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)\ ,$$

$$\alpha(t) = \sum_{v=0}^{\infty} p_v \cos vt\ , \quad \beta(t) = \sum_{v=0}^{\infty} p_v \sin vt\ ,$$

$$\alpha_n = \int_0^{\pi} \phi(t)\alpha(t) \cos nt\, dt\ , \quad \beta_n = \int_0^{\pi} \phi(t)\beta(t) \sin nt\, dt\ ,$$

$$w(\delta) = \sup_{0 \leq |t| \leq \delta} |f(x+t) - f(x)|\ .$$

p and q are mutually conjugate indices in the sense that $1/p + 1/q = 1$.

Recently M. Izumi and S. Izumi ([2, Th. 3]) proved the following

THEOREM A. *Let $\{p_n\}$ be a positive decreasing and convex sequence tending to zero and satisfying the condition*

$$\sum_{n=1}^{\infty} p_n^p n^{p-2} < \infty \, , \qquad (1 < p \leqq 2) \, .$$

If the modulus of continuity $\omega(\delta)$ of f satisfies the conditions

$$\sum_{n=1}^{\infty} \frac{\omega\left(\frac{1}{n}\right)}{n^{1/q} P_n} < \infty \, ,$$

and

$$\sum_{m=n}^{\infty} \frac{1}{m^p\left(\omega\left(\frac{1}{m}\right)\right)^{p-1}} \leqq \frac{C}{\left(n\omega\left(\frac{1}{n}\right)\right)^{p-1}} \tag{2.1}$$ [1]

then the Fourier series of f is $|N, p_n|$ summable.

In this note we prove that the condition (2.1) of the above theorem is *redundant* in that the assertion of the theorem holds without the condition (2.1) as well. The final result is then embodied in the following

THEOREM. *Let $\{p_n - p_{n+1}\}$ be a nonincreasing sequence and*

$$\sum_{n=1}^{\infty} p_n^p n^{p-2} < C \, , \qquad (1 < p \leqq 2) \, . \tag{2.2}$$

If the modulus of continuity of the continuous function $f(x)$ satisfies the condition

$$\sum_{n=1}^{\infty} \omega(n^{-1}) P_n^{-1} n^{-1/q} < C \, , \tag{2.3}$$

then the Fourier series of f is $|N, p_n|$ summable.

It is known that (see [4, Chapter XII, proof of Lemma 6.6]) the condition (2.2) of the theorem implies that

$$\sum_{n=1}^{\infty} P_n^p n^{-2} < C \, .$$

Also it is easy to show that the above condition implies the condition (2.2) of the theorem. Since p_n is nonnegative and nonincreasing

[1] Throughout the paper C denotes a positive constant, not necessarily the same at each occurrence.

we have $np_n \leqq P_n$ and therefore

$$\sum_{n=1}^{\infty} p_n^p n^{p-2} \leqq \sum_{n=1}^{\infty} P_n^p n^{-2} .$$

Thus the conditions (2.2) and $\sum_{n=1}^{\infty} P_n^p n^{-2} < C$ are equivalent. In view of this equivalence it follows that the theorem established here generalises an earlier result of the author [3] as well.

3. The following lemmas are required for the proof of the theorem.

LEMMA 1. *Under the condition* (2.2) *of the theorem*

$$\int_0^{1/n} \omega(t)P(t^{-1})dt \leqq C\omega(n^{-1})n^{-1/q} .$$

Proof. Remembering that the condition (2.2) of the theorem implies that

$$\sum_{n=1}^{\infty} P_n^p n^{-2} < C ,$$

we have

$$\begin{aligned}\int_0^{1/n} \omega(t)P(t^{-1})dt &\leqq \sum_{v=n}^{\infty} \omega(v^{-1})P_v v^{-2} \\ &\leqq \left[\sum_{v=n}^{\infty} \{\omega(v^{-1})v^{-2+2/p}\}^q\right]^{1/q}\left[\sum_{v=n}^{\infty} P_v^p v^{-2}\right]^{1/p} \\ &\leqq C\omega(n^{-1})n^{1/p-1} ,\end{aligned}$$

which is equivalent to the assertion of the lemma.

LEMMA 2. ([4, Chapter XII, Lemma 6.6]). *For the function* $\alpha(t)$ *to belong to the class* $L^p(p > 1)$ *it is necessary and sufficient that the condition* (2.2) *of the theorem is satisfied.*

LEMMA 3. ([1, Lemmas 5.11, 5.14 and 5.32]). *If* p_n *is nonnegative and nonincreasing, then for* $0 \leqq a < b \leqq \infty$, $0 < t \leqq \pi$ *and any* n

$$\left|\sum_{v=a}^{b} p_v e^{i(n-v)t}\right| \leqq CP(t^{-1}) , \tag{3.1}$$

$$\sum_{v=n}^{\infty} \frac{v(p_v - p_{v+1})}{P_v P_{v-1}} \leqq CP_{n-1}^{-1} , \tag{3.2}$$

and

$$P(2^\lambda) \leqq CP(2^{\lambda-1}) , \tag{3.3}$$

as $\lambda \to \infty$.

LEMMA 4. ([3, Lemma 5.20]). *If* p_n *is nonnegative and nonincreasing and if we take*

$$\gamma(t) = \sum_{v=0}^{\infty} p_v e^{ivt}$$

then for t *in* (h, π)

$$|\gamma(t + 2h) - \gamma(t)| \leqq Cht^{-1}P(h^{-1}) .$$

LEMMA 5. ([1, see proof of Lemma 5.16]). *If* p_n *is nonnegative and nonincreasing,* $\lim_{n\to\infty} p_n = 0$ *and* $\{p_n - p_{n+1}\}$ *is nonincreasing, then*

$$\begin{aligned}&\frac{1}{P_{n-1}}\left|\int_{1/n}^{\pi}\phi(t)\left\{\sum_{v=n}^{\infty} p_v \cos(n-v)t + \sum_{v=0}^{n-1}\frac{p_n}{P_n}P_v\cos(n-v)t\right\}dt\right| \\ &\leqq C\frac{p_n}{P_nP_{n-1}} + C\left[\frac{n(p_n - p_{n-1})}{P_nP_{n-1}} + \frac{p_n}{P_nP_{n-1}}\right]\int_{1/n}^{\pi}|\phi(t)|\,P(t^{-1})t^{-1}dt .\end{aligned}$$

LEMMA 6. ([2]). *Under the conditions* (2.2) *and* (2.3) *of the theorem*

$$\sum_{n=1}^{\infty}\frac{\omega\left(\frac{1}{n}\right)}{n} \leqq C .$$

4. *Proof of the theorem.* For the Fourier series

$$S_v(x) - f(x) = \frac{1}{\pi}\int_0^{\pi}\phi(t)\left(\frac{1}{2} + \sum_{k=1}^{v}\cos kt\right)dt ,$$

so that from (1.1) and Abel's transformation we have

$$\begin{aligned}&\pi\,|t_n - t_{n-1}| \\ &= \left|\int_0^{\pi}\phi(t)\left\{\sum_{v=0}^{n-1}\left(\frac{P_{n-v-1}}{P_n} - \frac{P_{n-v-2}}{P_{n-1}}\right)\cos(v+1)t\right\}dt\right| \\ &= \frac{1}{P_nP_{n-1}}\left|\int_0^{\pi}\phi(t)\sum_{v=0}^{n-1}(p_vP_n - p_nP_v)\cos(n-v)t\,dt\right| \\ &= \left|\frac{1}{P_{n-1}}\int_0^{\pi}\phi(t)\left(\sum_{v=0}^{\infty}p_v\cos(n-v)t\right)dt\right. \\ &\quad\left. - \frac{1}{P_nP_{n-1}}\int_0^{\pi}\phi(t)\left(\sum_{v=n}^{\infty}p_vP_n\cos(n-v)t + \sum_{v=0}^{n-1}p_nP_v\cos(n-v)t\right)dt\right|\end{aligned} \tag{4.1}$$

$$\leqq \frac{1}{P_{n-1}}\left|\int_0^\pi \phi(t)\alpha(t)\cos nt\,dt\right| + \frac{1}{P_{n-1}}\left|\int_0^\pi \phi(t)\beta(t)\sin nt\,dt\right|$$
$$+ \frac{1}{P_{n-1}}\left|\int_0^{1/n} \phi(t)\sum_{v=n}^{\infty} p_v \cos(n-v)t\,dt\right|$$
$$+ \frac{p_n}{P_n P_{n-1}}\left|\int_0^{1/n} \phi(t)\sum_{v=0}^{n-1} P_v \cos(n-v)t\,dt\right|$$
$$+ \frac{1}{P_{n-1}}\left|\int_{1/n}^{\pi} \phi(t)\left\{\sum_{v=n}^{\infty} p_v\cos(n-v)t + \sum_{v=0}^{n-1}\frac{p_n}{P_n}P_v\cos(n-v)t\right\}dt\right| .$$
$$= \sum_{r=1}^{5} |x_n^{(r)}| , \quad \text{say} .$$

From (4.1) and the definition of the absolute Nörlund summability it is clear that for establishing the theorem we have to prove that

$$\sum_{n=2}^{\infty} |x_n^{(r)}| < \infty , \qquad (r = 1, 2, \cdots, 5) . \tag{4.2}$$

Now

$$\begin{aligned} \sum_{n=2}^{\infty} |x_n^{(1)}| &= \sum_{\lambda=1}^{\infty} \sum_{n=2^{\lambda-1}+1}^{2^\lambda} |\alpha_n| P_{n-1}^{-1} \\ &\leqq \sum_{\lambda=1}^{\infty}\Big(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |\alpha_n|^q\Big)^{1/q}\Big(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} P_{n-1}^{-p}\Big)^{1/p} \\ &\leqq C\sum_{\lambda=1}^{\infty} 2^{\lambda/p}P^{-1}(2^\lambda)\Big(\sum_{n=1}^{\infty}\Big|\alpha_n \sin\frac{n\pi}{2^{\lambda+1}}\Big|^q\Big)^{1/q} \end{aligned} \tag{4.3}$$

making use of (3.3) of Lemma 3.

Since the function $\phi(t)$ is bounded in $[0, \pi]$ and by Lemma 2, under the condition (2.2) of the theorem, $\alpha(t) \in L^p$, it follows that $\phi(t)\alpha(t) \in L^p$. Also, it is known [1] that the Fourier series of $\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)$ is $-4/\pi \sum \alpha_n \sin nt \sin nh$, and therefore by Hausdorff-Young inequality we get

$$\begin{aligned} &\Big(\sum_{n=1}^{\infty} |\alpha_n \sin nh|^q\Big)^{p/q} \\ &\leqq C\int_0^\pi |\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)|^p dt \\ &\leqq C\int_0^\pi |\phi(t+h) - \phi(t-h)|^p\,|\alpha(t+h)|^p dt \\ &\quad + C\int_0^\pi |\alpha(t+h) - \alpha(t-h)|^p\,|\phi(t-h)|^p dt \\ &\leqq C\omega^p(h)\int_0^\pi |\alpha(t+h)|^p dt + C\int_{-h}^{\pi-h} |\alpha(t+2h) - \alpha(t)|^p\,|\phi(t)|^p dt \\ &\leqq C\omega^p(h) + C\int_{-h}^{h} \omega^p(|t|)\,|\alpha(t+2h)|^p dt \end{aligned} \tag{4.4}$$

$$+ C\int_{-h}^{h}\omega^p(|t|)\,|\alpha(t)|^p dt + C\int_{h}^{\pi}|\alpha(t+2h)-\alpha(t)|^p\omega^p(t)dt$$
$$\leqq C\omega^p(h) + Ch^pP^p(h^{-1})\int_h^{\pi}\omega^p(t)t^{-p}dt$$

using Lemma 4 and remembering that by virtue of Lemma 2, $\alpha(t)\in L^p$.

Taking $h=\pi/2^{\lambda+1}$ in the estimate (4.4) and then substituting it in (4.3) we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}|x_n^{(1)}| \\
&\leqq C\sum_{\lambda=1}^{\infty}2^{\lambda/p}P^{-1}(2^\lambda)\Big[\omega^p\Big(\frac{\pi}{2^{\lambda+1}}\Big)+2^{-\lambda p}P^p(2^\lambda)\int_{\pi/2^{\lambda+1}}^{\pi}\omega^p(t)t^{-p}dt\Big]^{1/p} \\
&\leqq C\sum_{\lambda=1}^{\infty}2^{\lambda/p}P^{-1}(2^\lambda)\omega\Big(\frac{\pi}{2^{\lambda+1}}\Big)+C\sum_{\lambda=1}^{\infty}2^{\lambda((1/p)-1)}\Big(\int_{\pi/2^{\lambda+1}}^{\pi}\frac{\omega^p(t)}{t^p}dt\Big)^{1/p} \\
(4.5)\quad &\leqq C\sum_{n=1}^{\infty}\omega\Big(\frac{\pi}{n}\Big)P_n^{-1}n^{-1/q}+C\sum_{\lambda=1}^{\infty}2^{\lambda((1/p)-1)}\Big\{\Big(\int_{1/\pi}^{1}+\int_{1}^{2^\lambda}\Big)\frac{\omega^p(t^{-1})}{t^{2-p}}dt\Big\}^{1/p} \\
&\leqq C+C\sum_{\lambda=1}^{\infty}2^{\lambda((1/p)-1)}\sum_{m=1}^{\lambda}\Big(\sum_{n=2^{m-1}+1}^{2^m}\omega^p(n^{-1})n^{p-2}\Big)^{1/p} \\
&\leqq C+C\sum_{m=1}^{\infty}2^{m(1-(1/p))}\omega(2^{-m})\sum_{\lambda=m}^{\infty}2^{\lambda((1/p)-1)} \\
&\leqq C+C\sum_{n=1}^{\infty}\omega(n^{-1})n^{-1}\leqq C\,,
\end{aligned}$$

by virtue of the condition (2.3) of the theorem and Lemma 6.

Similarly, we can prove that

$$(4.6)\qquad \sum_{n=2}^{\infty}|x_n^{(2)}|<\infty\,.$$

Also,

$$(4.7)\qquad \begin{aligned}\sum_{n=2}^{\infty}|x_n^{(3)}| &\leqq C\sum_{n=2}^{\infty}P_{n-1}^{-1}\int_0^{1/n}\omega(t)P(t^{-1})dt \\ &\leqq C\sum_{n=1}^{\infty}\omega(n^{-1})P_n^{-1}n^{-1/q}\leqq C\,,\end{aligned}$$

by the application of (3.1) of Lemma 3, Lemma 1 and the condition (2.3) of the theorem.
For the proof of

$$(4.8)\qquad \sum_{n=2}^{\infty}|x_n^{(4)}|<C\,,$$

see the proof of $\sum_{n=1}^{\infty}K_n<\infty$ in [2]. Finally, by Lemma 5 we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty} | x_n^{(5)} | \\
&\leqq C \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} + C \sum_{n=1}^{\infty} \left[\frac{n(p_n - p_{n+1})}{P_n P_{n-1}} + \frac{p_n}{P_n P_{n-1}} \right] \int_{1/n}^{\pi} \omega(t) P(t^{-1}) t^{-1} dt \\
&\leqq C \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} + C \sum_{n=1}^{\infty} \frac{n(p_n - p_{n+1})}{P_n P_{n-1}} \\
&\quad + C \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \omega(v^{-1}) P_v v^{-1} \\
&\quad + C \sum_{n=1}^{\infty} \frac{n(p_n - p_{n+1})}{P_n P_{n-1}} \sum_{v=1}^{n} \omega(v - 1) P_v v^{-1} \qquad (4.9) \\
&\leqq C + C \sum_{v=1}^{\infty} \omega(v^{-1}) P_v v^{-1} \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}} \\
&\quad + C \sum_{v=1}^{\infty} \omega(v^{-1}) P_v v^{-1} \sum_{n=v}^{\infty} \frac{n(p_n - p_{n+1})}{P_n P_{n-1}} \\
&\leqq C + C \sum_{v=1}^{\infty} \omega(v^{-1}) v^{-1} \leqq C \, ,
\end{aligned}
$$

by the application of the estimate (3.2) of Lemma 3 and Lemma 6.

Combining the estimates in (4.5) – (4.9) we find that (4.2) is established. This completes the proof of the theorem.

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BANARAS HINDU UNIVERSITY
VARANASI-5, INDIA