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INVARIANT SUBSPACES AND PROJECTIVE REPRESENTATIONS

KEITH YALE

Let Γ be a subgroup of the real line R with the discrete topology, and let G be its compact dual group. This paper shows the existence of a (nontrivial) simply invariant subspace of $L^2(G)$ which is not of the form $\varphi H^2(G)$ provided Γ contains at least two rationally independent elements. The proof relies heavily on the existence of a nontrivial local projective representation of the two-dimensional torus.

Helson and Lowdenslager [4] showed the existence of a simply invariant subspace not of the form $\varphi H^2(G)$ in case Γ contains an infinite set of rationally linearly independent elements. We use the correspondence introduced in [4] between simply invariant subspaces and cocycles but in contrast to [4] we use nontrivial local projective multipliers to show that the appropriate cohomology group is nontrivial.

The connection between invariant subspaces and cocycles is discussed in § 2 and in § 3 we will give a quotient group argument which allows us to reduce the general problem to its specialization on the two-dimensional torus. Sections 4 and 5 relate the notion of projective representation with a cocycle and it is shown that a nontrivial projective representation gives rise to a cocycle whose corresponding subspace is not of the form $\varphi H^2(G)$.

2. Preliminaries. Let G be an arbitrary locally compact Abelian group dual to Γ and let Λ be a continuous one-parameter subgroup of G which we also denote by $\{e_t \mid t \text{ in } R\}$. Haar measure in G will be denoted by dx and will be normalized to have total mass one in case G is compact. As usual, a.e. (x) means for all but a set of Haar measure zero. A (Borel) function φ on G is said to be *unitary* in case $\varphi(x)$ has modulus one a.e. (x).

DEFINITION. A function A on $\Lambda \times G$ is said to be a *cocycle* on G in case:

(2.1) $\qquad A(e_t, \quad)$ is a unitary function for each e_t in Λ,

(2.2) $\qquad A(e_t + e_u, x) = A(e_u, x)A(e_t, x - e_u)$ for all e_t, e_u in Λ

and a.e. (x), and

(2.3) $\qquad \Lambda$ is strongly continuous in the sense that $A(e_t, \quad)f$

is a continuous function from R into $L^2 = L^2(G)$ for f in L^2.

Cocycles of the form

$$A(e_t, x) = \varphi(x)/\varphi(x - e_t), \quad \text{all } e_t \text{ in } \Lambda, \text{ a.e. } (x) \tag{2.4}$$

for some unitary function φ are called *coboundaries.* We will frequently denote $A(e_t, x)$ by $A(t, x)$.

If λ is in Γ we let χ_λ be the character on G defined by $\chi_\lambda(x) = x(\lambda)$ for all x in G; the corresponding unitary representation V_0 of Γ is given by

$$V_0(\lambda)f(x) = \chi_\lambda(x)f(x) \tag{2.5}$$

for all f in L^2. Any bounded operator on L^2 which commutes with all the $V_0(\lambda)$ is necessarily a multiplication by a function in L^∞. Let U_0 be the unitary representation of G defined by

$$U_0(x)f(y) = f(y - x) \tag{2.6}$$

for all f in L^2.

For the remainder of this section we will let Γ be a subgroup of the real line R. Let G be the compact Abelian group dual to the discretely topologized Γ. A closed subspace $\mathscr{M}$ of L^2 is said to be *simply invariant* in case $V_0(\lambda)\mathscr{M} \subsetneqq \mathscr{M}$ if and only if $\lambda \geqq 0$. The Hardy space H^2 consists of those functions f in L^2 whose Fourier transforms $\hat{f}(\lambda) = \int \chi_{-\lambda}(x)f(x)dx$ vanish for $\lambda < 0$. Subspaces of the form $\mathscr{M} = \varphi H^2 = \{\varphi f: f \text{ in } H^2\}$ where φ is a unitary function are simply invariant and in the case where G is a circle all simply invariant subspaces are of this form.

In order to avoid the rather special circle group we will henceforth suppose that Γ is dense in R. The characters e_t defined by $e_t(\lambda) = \exp(it\lambda)$ are distinct and provide a continuous one-parameter dense subgroup Λ of G. A correspondence is exhibited in [3, 4] between simply invariant subspaces $\mathscr{M}$ (suitably normalized) and cocycles A in such a way that $\mathscr{M} = \varphi H^2$ if and only if A is the coboundary (2.4). We therefore wish to construct cocycles which are not coboundaries.

If A is a coboundary then A can be extended from $\Lambda \times G$ to $G \times G$ so that (2.4) remains valid with t replaced by an arbitrary y in G and conversely. Moreover, the multiplication operator $A(y, \ )$ is the strong operator limit of a sequence $A(t_n, \ )$ where e_{t_n} tends to y in G; this observation will be useful later. Equivalently, A is a coboundary if and only if the unitary representation $U(t) = A(t, \ )U_0(t)$ can be extended from Λ to a (strongly continuous) unitary representation of G. A cocycle was constructed in [4] (in case Γ is suitably large) for which the unitary representation did not extend to G. However, it is conceivable that $U(t)$ might extend to a (local) projective representation of G; this idea is turned around and will be used to extract cocycles from projective representations.

There is a superficial answer to our problem in case Γ is not all of R for then there are trivial cocycles which are not coboundaries. For example, let $A(t, x) = \exp(-it\lambda)$ for some fixed real λ *not* in Γ. If λ *were* in Γ then A would be the coboundary with unitary function χ_λ but with λ not in Γ there is no unitary function φ such that $\exp(-it\lambda) = \varphi(x)/\varphi(x - e_t)$. Conversely, if A is a cocycle which is constant a.e. (x) for each t (the null set depending upon t), then $A(t, x) = \exp(-it\lambda)$, a.e. (x) for some fixed λ in R. We will call cocycles of this form *constant cocycles*. Consequently the nontrivial problem [3, p. 149] is to find cocycles which are not products of constant cocycles and coboundaries.

The cocycles defined in [3] were measurable functions on $\Lambda \times G$ but we will have no need for cocycles to be product measurable. Anyway, one can pass from one version to another [3, p. 145], [2]. Also we have departed from [3] by making an insignificant sign change in our definition of cocycle.

3. Reduction to the torus. Suppose that $\Gamma_0 \subseteqq \Gamma$ are subgroups of the discrete real line and let G_0 and G be their compact dual groups. To each cocycle A_0 on G_0 we will associate a cocycle A on G in such a way that if A is the product of a constant cocycle and a coboundary then so is A_0. Since the two-dimensional torus T^2 is dual to the group of lattice points Z^2 and Z^2 is isomorphic to a subgroup $\Gamma_0 \subseteqq \Gamma$ of any group $\Gamma \subseteqq R$ with at least two independent elements it will be sufficient to construct a cocycle on T^2 which is not the product of a constant cocycle and a coboundary.

Define a closed subgroup $H = \{x \text{ in } G \mid \chi_\lambda(x) = 1 \text{ for all } \lambda \text{ in } \Gamma_0\}$ of G so that G_0 can be identified with G/H. Let π be the usual quotient map from G onto G/H and let e_t and ε_t be the previously defined one-parameter groups Λ and Λ_0 in G and G_0. One can verify $\pi(e_t) = \varepsilon_t$ by noting that ε_t is the restriction of e_t from Λ to Λ_0.

If A_0 is a cocycle on G_0 we define a cocycle A on G by

$$A(e_t, x) = A_0(\varepsilon_t, \pi(x)) \tag{3.1}$$

for all (e_t, x) in $\Lambda \times G$.

For each t in R the measurable function $A(e_t, \ )$ on G is certainly unitary because $\pi^{-1}(S)$ is a null set in G whenever S is a null set in G/H. The cocycle identity (2.2) is easy enough to verify with the aid of $\pi(e_t) = \varepsilon_t$ so all that remains is the strong continuity.

Let the Haar measures dx and dx_0 in G and G/H both be normalized to have total mass one. There is a normalization for the Haar measure $d\xi$ on H such that

$$\int_G f(x)dx = \int_{G/H}\left(\int_H f(x+\xi)d\xi\right)dx_0 \tag{3.2}$$

for all f in $L^1(G)$.

Let f be in $L^2(G)$ and put

$$g(x_0) = \int_H |f(x+\xi)|^2 d\xi$$

where $x_0 = \pi(x)$. A straight-forward computation with (3.2) shows that $A(e_t,\ )f$ moves continuously in $L^2(G)$ as t varies because $A_0(e_t,\ )\sqrt{g}$ moves continuously in $L^2(G/H)$.

THEOREM. *If A is the product of a constant cocycle and a coboundary then so is A_0.*

Proof. For some constant cocycle C and some unitary function φ on G we have

$$C(t)A(t, x) = \varphi(x)/\varphi(x - e_t)$$

for each real t and almost all x.

It is advantageous to normalize by choosing λ in R such that $\int \chi_\lambda(x)\varphi(x)dx$ does not vanish and putting $\psi = \chi_\lambda\varphi$. The cocycle $B = \chi_\lambda CA$ is really the coboundary.

$$B(t, x) = \psi(x)/\psi(x - e_t) \tag{3.3}$$

and we have $B_0 = \chi_\lambda CA_0$. Consequently it is sufficient to show that B_0 is a coboundary and we will do this by arguing that ψ must be constant on cosets of H.

Since $B(t, x) = B_0(t, \pi(x))$ it follows that $B(t, x) = B(t, x + h)$ for all real t and all (x, h) in $G \times H$. Now the coboundary B can be extended to $G \times G$ and, in fact, $B(y,\ )$ is a limit in $L^2(G)$ of a sequence $B(t_n,\ )$ where e_{t_n} goes to y in G. Therefore, passing to a subsequence if necessary, $B(t_n, x)$ tends to $B(y, x)$ for almost all x and we can conclude

$$B(y, x) = B(y, x + h) \tag{3.4}$$

for all y in G, h in H and almost all x in G.

From (3.3) (valid now for t replaced by any element in G) and (3.4) we have

$$\psi(x + \xi) = B(h, x)\psi(x + \xi - h) \tag{3.5}$$

for every ξ in H and almost all x in G. Integrating this last expression with respect to Haar measure $d\xi$ on H we find

$$\int_H \psi(x+\xi)d\xi = B(h, x)\int_H \psi(x+\xi)d\xi .$$

Now $\int_H \psi(x+\xi)d\xi$ does not vanish since $\int_G \psi(x)dx$ is not zero (consider (3.2)) and so we may conclude $B(h, x) = 1$ for all h in H and almost all x in G.

It follows from (3.5) that ψ is constant on cosets of H and so we can define a unitary function ψ_0 on G/H by $\psi_0(\pi(x)) = \psi(x)$. Clearly B_0 is a coboundary determined by ψ_0. That completes the proof.

4. **Projective representations and projective cocycles.** Let G be a locally compact Abelian group. A strongly continuous function U from G into the unitary operators on some Hilbert space is said to be a *projective representation* if

$$U(x)U(y) = \omega(x, y)U(x+y) \tag{4.1}$$

for some function ω of modulus one and if $U(0) = 1$. We say that ω is the *multiplier* of the representation and it is not difficult to show that it satisfies the identity $\omega(x, y)\omega(x+y, z) = \omega(y, z)\omega(x, y+z)$ and the normalizing condition $\omega(x, 0) = \omega(0, x) = 1$. Moreover, ω is continuous on $G \times G$. Conversely, given a function ω with these properties one can construct a projective representation U_ω with multiplier ω. Indeed, define U_ω on L^2 by

$$U_\omega(x)f(y) = \omega(x, y-x)f(y-x) . \tag{4.2}$$

The projective representation U_ω is of the form

$$U_\omega(x) = A_\omega(x, \ )U_0(x) \tag{4.3}$$

where $A_\omega(x, y) = \omega(x, y-x)$ is a function of modulus one on $G \times G$. The (projective) group property of U_ω implies that

$$\omega(x, y)A_\omega(x+y, z) = A_\omega(x, z)A_\omega(y, z-x) \tag{4.4}$$

and the strong continuity of U_ω implies that $A_\omega(x, \ )$ is a strongly continuous operator valued function in x.

Observe that A_ω differs from the ordinary cocycle (§ 2) in two respects; first, A_ω is a function on $G \times G$ instead of merely on $\Lambda \times G$, and, secondly, (4.4) replaces (2.2). We say that A_ω is a *projective cocycle*.

We say that ω is trivial if

$$\omega(x, y) = p(x)p(y)/p(x+y) \tag{4.5}$$

for some continuous function p of modulus one on G. In this case any projective representation U with multiplier ω can be made into an ordinary representation merely by multiplying $U(x)$ by $p(x)$. The product of two multipliers is again a multiplier and two multipliers whose quotient is trivial are said to be *equivalent*.

If ω and σ are equivalent multipliers so that

$$\omega(x, y)/\sigma(x, y) = p(x)p(y)/p(x + y) \tag{4.6}$$

then a direct computation will give

$$A_\omega(x, y)/A_\sigma(x, y) = p(x)(\varphi(y)/\varphi(y - x)) \tag{4.7}$$

where $\varphi(y) = 1/p(y)$. In particular if ω is trivial then A_ω is p times a coboundary and conversely.

Now suppose that G has a continuous one-parameter subgroup $\Lambda = \{e_t \mid t \in R\}$ and let A_ω be a projective cocycle on G with U_ω the corresponding projective representation as given by (4.3). We wish to extract an ordinary cocycle A from A_ω in such a way that A will not be the product of a constant cocycle and a coboundary if ω is a nontrivial multiplier.

Restrict U_ω to Λ so that it is a projective representation of the reals. It follows that (see the last paragraph of this section) U_ω is equivalent to an ordinary representation U given by

$$U(e_t) = p(e_t)U_\omega(e_t) \tag{4.8}$$

where

$$\omega(e_t, e_u) = p(e_t)p(e_u)/p(e_t + e_u) \tag{4.9}$$

for some continuous function p on Λ and for all $e_t, e_u \in \Lambda$. Observe that U satisfies the Weyl commutation relation

$$U(e_t)V_0(\lambda) = \chi_\lambda(-e_t)V_0(\lambda)U(e_t) \tag{4.10}$$

because U_ω does.

Consequently the operator $U(e_t)U_0(-e_t)$ commutes with all the $V_0(\lambda)$ so that

$$U(e_t) = A(e_t, \ )U_0(e_t), \ e_t \in \Lambda\ , \tag{4.11}$$

for some ordinary cocycle A.

From (4.8) and (4.11) we see that

$$A(e_t, x) = p(e_t)A_\omega(e_t, x) \tag{4.12}$$

for all $e_t \in \Lambda$ and a.e. (x).

We say that A is the cocycle induced by A_ω; it is uniquely determined up to a constant cocycle factor. If A is the product of a constant cocycle $e^{it\lambda}$ and a coboundary $\varphi(x)/\varphi(x - e_t)$ then (4.12) and (4.7) imply that ω is trivial.

This analysis will have to be refined to yield the desired result on the torus T^2 for T^2 has no nontrivial multipliers. However, there are $\frac{1}{2}n(n-1)+1$ inequivalent *local* multipliers on T^n or R^n as shown by Bargmann [1] and local multipliers are sufficient for our purposes. Notice, in particular, that R has no nontrivial local projective representations.

5. Local multipliers and cocycles on T^2. A *local projective* multiplier ω on the torus T^2 is a continuous function on some neighborhood $\mathscr{N} \times \mathscr{N}$ of the identity in $T^2 \times T^2$ which satisfies the same functional equation and normalizing condition as a multiplier whenever x, y and $x+y$ belong to $\mathscr{N}$. Unfortunately (4.3) cannot be used to define a local projective representation U_ω, or, equivalently, a local projective cocycle A_ω. We must resort to an *ad hoc* construction of U_ω starting from a specific nontrivial local projective multiplier ω. We can then extract a cocycle from U_ω in much the same manner as in § 4 and it is a matter of detail to prove that A is not the product of a constant cocycle and a coboundary.

Let T^2 be realized as the square $[-\pi, \pi] \times [-\pi, \pi]$ with the opposite edges identified and let $\mathscr{N}$ be the open neighborhood $(-\pi, \pi) \times (-\pi, \pi)$ of the identity. For a one-parameter subgroup Λ we will take the familiar winding line with irrational slope α.

Define ω on $\mathscr{N} \times \mathscr{N}$ by

$$\omega(x, y) = \exp i((x_2 - \alpha x_1)y_1 - (y_2 - \alpha y_1)x_1) \\ = \exp i(x_2 y_1 - y_2 x_1) \tag{5.1}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $-\pi < x_i, y_i < \pi$. This is the canonical example of a nontrivial local projective multiplier on T^2 [1].

Since the complement of $\mathscr{N}$ is a null set we can regard $\omega(x, \ )$ as a unitary function on T^2 for each fixed $x \in \mathscr{N}$. Now put $A_\omega(x, y) = \omega(x, y - x)$ whenever $x \in \mathscr{N}$ and $y \in \mathscr{N} + x$. Then $A_\omega(x, \ )$ is a unitary function on T^2 for each fixed $x \in \mathscr{N}$ (the exceptional null set depends upon x). For $x \in \mathscr{N}$ we define the unitary operator $U_\omega(x)$ by

$$U_\omega(x) = A_\omega(x, \ ) U_0(x) . \tag{5.2}$$

It is easily verified that U_ω is a strongly continuous operator valued function on $\mathscr{N}$.

We will now extract a cocycle A from A_ω even though $A_\omega(x, \ )$ is *not* defined for all x. The discussion parallels that of § 4 and will only be given in outline.

Let Λ_1 denote the connected segment of $\Lambda \cap \mathscr{N}$ (relative to the ordinary real line topology on Λ) which contains the identity and choose a proper segment Λ_0 of Λ_1 such that $0 \in \Lambda_0 \subsetneqq \Lambda_0 + \Lambda_0 \subsetneqq \Lambda_1$.

For $x, y \in \Lambda_0$, U_ω satisfies (4.1) so that U_ω is a local projective representation of the reals. Consequently U_ω is equivalent to a local ordinary representation U; this means that equations (4.8) and (4.9) hold for some continuous function p on Λ_1 (say) and for all $e_t, e_u \in \Lambda_0$. The local representation U can be extended to a representation U (keeping the same notation) of Λ [5, Th. 63] which must satisfy the Weyl commutation relation (4.10). Exactly as before we have $U(e_t) = A(e_t, \ )U_0(e_t)$, $e_t \in \Lambda$, for some ordinary cocycle A. We say that A is the cocycle induced by A_ω; notice that

$$A(e_t, x) = p(e_t)A_\omega(e_t, x) \tag{5.3}$$

holds only for $e_t \in \Lambda_0$, a.e. (x).

We will now show that A is not the product of a constant cocycle C and a coboundary. If, on the contrary, A is such a product, then

$$A_\omega(e_t, x) = \bar{p}(e_t)(\varphi(x)/\varphi(x - e_t)) \tag{5.4}$$

holds for all $e_t \in \Lambda_0$, a.e. (x) where we have relabeled the continuous function C/p on Λ_1 by $\bar{p}$. In terms of the unitary operators U_ω and $U(y) = (\varphi(\ \ )/\varphi(\ \ - y))U_0(y)$, $y \in T^2$, equation (5.4) becomes $U_\omega(e_t) = \bar{p}(e_t)U(e_t)$ for all $e_t \in \Lambda_0$.

We wish to extend p from Λ_0 to $\Lambda \cap \mathscr{N}$ in such a way that (5.4) remains valid. A continuity argument will then enable us to extend p from $\Lambda \cap \mathscr{N}$ to $\mathscr{N}$ and this will imply that ω is trivial.

To extend p from Λ_0 to $\Lambda \cap \mathscr{N}$ let $y \in \Lambda \cap \mathscr{N}$ so that $y \in M\Lambda_0 = \{Me_t \mid e_t \in \Lambda_0\}$ for some integer $M > 0$. Thus $e_t = yM \in \Lambda_0$ and suppose, for the moment, that $ne_t \in \mathscr{N}$ for all $n \leqq M$. Then

$$\begin{aligned} U(y) &= U(Me_t) = (U(e_t))^M \\ &= (p(e_t)U_\omega(e_t))^M \\ &= [(p(e_t))^M \textstyle\prod_{k=1}^{M-1} \omega(e_t, \ \ (M-k)e_t)]U_\omega(y) \end{aligned}$$

and we can define $p(y)$ to be the value of the expression in the brackets which obviously is independent of the representation $y = Me_t$.

This definition of $p(y)$ is valid whenever $(M-k)e_t$ is in the domain of $\omega(e_t, \ )$, i.e., whenever $ne_t \in \mathscr{N}$ for all $0 \leqq n \leqq M$. For each M there are only finitely many $y \in M\Lambda_0$ such that $ne_t \notin \mathscr{N}$ for some $0 \leqq n \leqq M$. For these exceptional values we can define $p(y)$ by continuity (relative to the usual real line topology on Λ) so that

$$U(y) = p(y)U_\omega(y) \tag{5.5}$$

holds for all $y \in \Lambda \cap \mathscr{N}$; or, equivalently, so that (5.4) holds for all e_t in $\Lambda \cap \mathscr{N}$.

To extend p from $\Lambda \cap \mathscr{N}$ to $\mathscr{N}$ we need only note that $\Lambda \cap \mathscr{N}$ is dense in $\mathscr{N}$. Let $y \in \mathscr{N}$ and choose a sequence $y_n \in \Lambda \cap \mathscr{N}$ which converges to y. Hence $p(y_n)I = U(y_n)U_\omega(-y_n)$ tends strongly to

$$U(y)U_\omega(-y)$$

and this limit must be of the form $p(y)I$. Alternately, $U(y)U_\omega(-y)$ is a multiple of the identity for each y in $\mathscr{N}$ because it commutes with all bounded operators when y varies over a dense subset of $\Lambda \cap \mathscr{N}$. We have now constructed a continuous function p on $\mathscr{N}$ such that (5.5) holds for all y in $\mathscr{N}$. Since U_ω is a nontrivial local projective representation of $\mathscr{N}$ this is a contradiction. Hence the induced cocycle A cannot be the product of a constant cocycle and a coboundary. That completes the proof.

An interesting question remains. If A is a cocycle on T^2 can one find a local projective cocycle A_ω which induces A? An affirmative answer should enable one to settle some of the open function theoretic questions on T^2.

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UNIVERSITY OF MONTANA
MISSOULA, MONTANA