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SUBADDITIVE FUNCTIONS

CHI SONG WONG

In a recent paper, D. W. Boyd and J. S. W. Wong ask for an example of positive subadditive function ϕ for which $\phi(t) < t$ for all t and there exist $0 < c < d$ such that $\sup_{t \in [c,d]} \phi(t)/t = 1$. Our main result is that such an example does not exist.

For a metric space (X, d), we shall use Ran d to denote the set $\{d(x, y): x, y \in X\}$ and use *cl* Ran d to denote the closure of Ran d. Let X be a complete metric space, let ψ be a function of *cl* Ran d into $[0, \infty)$ and let T be a function of X into itself such that T is ψ-contractive on X $(d(T(x), T(y)) \leqq \psi(d(x, y)), x, y \in X)$. E. Rakotch [3, Corollary to Theorem 2] shows that if there is a decreasing function α on $[0, \infty)$ such that for any $t > 0$, $\alpha(t) < 1$ and $\psi(t) = \alpha(t)t$, then T has a unique fixed point. In order to show that Theorem 2 in [1] actually extends the above result when X is metrically convex, D. W. Boyd and J. S. W. Wong [1, p. 464] ask for an example of positive subadditive function ϕ for which $\phi(t) < t$ for all $t > 0$ and there exist $0 < c < d$ such that

$$\sup_{t \in [c,d]} \phi(t)/t = 1 .$$

We now show that such an example does not exist.

THEOREM. *Let ϕ be a positive subadditive function on an interval $(0, b)$ $(0 < b \leqq \infty)$ such that $\phi(t) < t$ for all t in $(0, b)$. Then*

$$\sup_{a \leqq t < b} \phi(t)/t < 1 , \qquad 0 < a < b .$$

Proof. If $b = \infty$, then by subadditivity and Theorem 7.6.2 in [2],

$$\lim_{t \to \infty} \phi(t)/t = \inf_{t > 0} \phi(t)/t ;$$

thus from $\phi(t)/t < 1$, $\sup_{t > a} \phi(t)/t < 1$ for large a's. So we may assume that $b < \infty$. Suppose to the contrary that there exist c, d in $(0, b)$ such that $c < d$ and

$$\sup_{t \in [c,d]} \phi(t)/t = 1 .$$

Then there exists a sequence $\{t_n\}$ in $[c, d]$ such that

$$(1) \qquad \lim_{n \to \infty} \phi(t_n)/t_n = 1 .$$

By compactness of $[c, d]$, we may, by taking a subsequence, assume that $\{t_n\}$ converges to some t in $[c, d]$. Let m be any positive integer. Then by the subadditivity of ϕ,

$$(2) \qquad \phi(t_n) \leqq m\phi(t_n/m) < t_n , \qquad n \geqq 1.$$

From (1) and (2),

$$(3) \qquad \lim_{n\to\infty} \phi(t_n/m) = t/m .$$

We now prove by induction that

$$(4) \qquad \lim_{n\to\infty} \phi(jt_n/2^k) = jt/2^k , \qquad k \geqq 1,\ 0 < j < 2^k,\ j \text{ is odd.}$$

Assume that (4) is true for $k \leqq i$, where i is given. Let j be any odd number in $(0, 2^{i+1})$. Then

$$(5) \qquad \phi((j+1)t_n/2^{i+1}) \leqq \phi(jt_n/2^{i+1}) + \phi(t_n/2^{i+1}) < (j+1)t_n/2^{i+1} .$$

By (5), the induction hypothesis and (3) (also by (1) if $j = (2^{i+1} - 1)/2^{i+1}$), we have by letting $n \to \infty$,

$$(j+1)t/2^{i+1} \leqq \limsup_{n\to\infty} \phi(jt_n/2^{i+1}) + t/2^{i+1} \leqq (j+1)t/2^{i+1} ,$$

i.e.,

$$\lim_{n\to\infty} \phi(jt_n/2^{i+1}) = jt/2^{i+1} ,$$

proving (4). Take any s in $(0, t)$. It suffices to prove that $s \leqq \phi(s)$. Since the set

$$D = \{jt/2^k \colon k \geqq 1,\ 0 < j < 2^k,\ j \text{ is odd}\}$$

is dense in $(0, t)$, there exists a strictly decreasing sequence $\{s_n\}$ *in* D which converges to s. By (4), there is a sequence $\{w_n\}$ for which

$$(6) \qquad s_n - 1/n < w_n < s_n,\ \phi(w_n) > s_n - 1/n , \qquad n \geqq 1.$$

Now

$$(7) \qquad \phi(w_n) \leqq \phi(w_n - s) + \phi(s) < (w_n - s) + \phi(s) , \qquad n \geqq 1.$$

From (6) and (7), we obtain $s \leqq \phi(s)$.

From the above result, we know that the condition (24) in [1] can be dropped. We thus have an improved version of [1, Proposition].

PROPOSITION. Let (X, d) be a complete metrically convex metric space and let $f\colon X \to X$. Suppose that there is a function ψ of cl Ran d into $[0, \infty)$ such that $\psi(t) < t$ for all t in cl Ran $d - \{0\}$ and f is ψ-contractive on X. Then there exists a decreasing function α on $[0, \infty)$

such that $\alpha(t) < 1$ for all $t > 0$ and

$$d(f(x), f(y)) \leqq \alpha(d(x, y))d(x, y)\ , \qquad x, y \in X.$$

D. W. Boyd and J. S. Wong show that α in the above proposition can actually be constructed as follows:

$$\alpha(t) = \sup_{s \geqq t} \phi(s)/s\ , \qquad t > 0\ ,$$

where

$$(*) \qquad \phi(t) = \sup\{d(f(x), f(y)) \colon x, y \in X,\ d(x, y) = t\}\ , \quad t \in \operatorname{Ran} d.$$

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SOUTHERN ILLINOIS UNIVERSITY