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ON STABLE FIBER SPACE OBSTRUCTIONS

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It will be proved that every stable fiber space obstruction is a coset of the image of some stable twisted cohomology operation.

1. Statement of the theorem. Consider the following tower of K-principal fibrations over a path connected space B.

$$\begin{array}{ccccccccc} & & C_i & & C_3 & & C_2 & & C_1 \\ & & \uparrow{\scriptstyle k_i} & & \uparrow & & \uparrow & & \uparrow{\scriptstyle k_1} \\ E_{i+1} & \longrightarrow & E_i & \cdots\longrightarrow & E_3 & \longrightarrow & E_2 & \longrightarrow & B\,. \end{array}$$

Each C_i is a product, over $K = K(\pi, 1)$, of $L(G, n)$'s. $\pi = \pi_1(B)$ and $L(G, n) = L_\phi(G, n)$ where $\phi: \pi \to \operatorname{aut} G$. ("$L_\phi(G, n)$" and "$K$-principal fibration" are discussed in § 2. They generalize "$K(G, n)$" and "principal fibration".) Suppose X is a CW space and $f: X \to B$ is a given map. Define

$$O_i(f) = \{\hat{f}^* k_i \,|\, \hat{f}: X \to E_i \text{ is a lifting of } f\} \subset [X, C_i]_K\,.$$

Then $O_1(f), O_2(f), \cdots, O_i(f)$, are the successive obstructions to lifting f to E_{i+1} (f lifts to E_{i+1} if and only if $0 \in O_j(f), 1 \leqq j \leqq i$). A more detailed description of local coefficient obstruction theory is given in [4]. The purpose of the present paper is to prove the following theorem.

THEOREM. *Assume that there is an N such that the C_j's are products of $L(G, n)$'s and $N + 1 \leqq n < 2N$. Then there is a stable B-operation $\Phi = \Phi_{x,f}: [X, \Omega C_1]_K \rightharpoonup [X, C_i]_K$ (an additive relation) such that $O_i(f)$ is a coset of the subgroup* Image $\Phi \subset [X, C_i]_K$.

It will be seen from the proof that $O_i(f)$ is a coset under more general circumstances. The hypotheses stated here cover the case of a modified Postnikov system for a fibration $F \to E \to B$ (not necessarily orientable) when $\dim X \leqq 2C$, $C =$ connectivity of F.

For simplicity, all spaces are assumed to be path connected, pointed and to have the homotopy type of CW spaces. All maps are assumed pointed.

The result here seems to be new even for orientable systems. Results similar to this have been obtained, independently and earlier, by Mahowald [2] for orientable sphere bundles and Meyer [5] for orientable fibrations. However, neither shows that the obstruction is actually a coset.

2. **Some definitions.** First, recall from [Gitler 1] and [Siegel 7] the definition of $L_\phi(G, n)$. Let $\phi; \pi \to \operatorname{aut} G$ be a homomorphism of a group π into the automorphism group of an abelian group G. Then there is an associated map $\bar{\phi}$ from π into the group of base point preserving homeomorphisms of $K(G, n)(K(G, n)$ is a pointed CW space which is an Eilenberg-MacLane space of type $(G, n))$. The following diagram is commutative

$$\begin{array}{ccc} \pi \times \pi_n(K(G, n), *) & \xrightarrow{\bar{\phi}*} & \pi_n(K(G, n), *) \\ \updownarrow & & \updownarrow \\ \pi \times G & \xrightarrow{\quad\phi\quad} & G \end{array}$$

where $\bar{\phi}_*(x, y)$ means $\bar{\phi}(x)_*(y)$. Let $\pi \to \hat{L} \to K$ be the universal cover of $K = K(\pi, 1)$. It is a universal principal π-bundle. Let $L_\phi(G, n) \to K(\pi, 1)$ be the associated bundle with fiber $K(G, n)$. It has a natural section. (L, K) can be assumed to have the homotopy type of a CW pair. Let Γ be a local coefficient system on a space X classified by $x: X \to K = K(\pi, 1)$. Then $H^n(X: \Gamma) \leftrightarrow [X, L]_K$ where the latter is the set of homotopy classes of maps over K (see [Steenrod, 8] or [Olum, 6]).

Now suppose, in general, that $D \xrightarrow{\check{\omega}} W \xrightarrow{\hat{\omega}} D$ and $\hat{\omega}\check{\omega} =$ identity. Define

$$\begin{aligned} \bar{P}W &= \{k \in W^I \mid \hat{\omega}k(t) = \hat{\omega}k(t'), t, t' \in I, k(0) = \hat{\omega}\check{\omega}(0)\} \\ \bar{\Omega}W &= \{k \in \bar{P}W \mid k(0) = k(1)\} . \end{aligned}$$

If $W \to D$ is a fibration then $\bar{P}W \to W(k \to k(1))$ is a fibration with fiber ΩW (the ordinary loop space). Call it and any fibration induced from it a *D-principal fibration*. The following diagram is a pullback.

$$\begin{array}{ccc} \bar{\Omega}W & \longrightarrow & \bar{P}W \\ \downarrow & & \downarrow \\ D & \longrightarrow & W . \end{array}$$

Now apply these definitions to $K(\pi, n) \to L_\phi(G, n) \to K(\pi, 1)$. $\bar{\Omega}L$ is $L_\phi(G, n-1)$. $f: X \to L$ lifts to $\bar{P}L$ if and only if f is zero as an element of $H^n(X; \Gamma)$ if and only if f factors up to homotopy through $K \to L$.

Next, define $L_D(G, n) = L_{D,\phi,\alpha}(G, n)$ as follows. Suppose $\alpha: \pi_1(D) \to \pi$ is a given homomorphism corresponding to a map $D \to K(\pi, 1)$ also denoted by α. Define $L_D(G, n)$ as the pullback of α and $L_\phi(G, n) \to K(\pi, 1)$. If the classifying map for a local coefficient system Γ on X can be factored through α then $H^n(X, \Gamma) \leftrightarrow [X, L_D(G, n)]_D$. Suppose $\pi_1(D) = \pi$, $\alpha =$ the classifying map for Γ, and ϕ determines Γ. Then write $L_\Gamma(G, n)$ for $L_{D,\phi,\alpha}(G, n)$.

3. Twisted operations with local coefficients. Let Γ and Γ' be local coefficient systems on a space D determined by $\phi\colon \pi \to \operatorname{aut} G$ and $\Psi\colon \pi \to \operatorname{aut} H$ respectively, where $\pi = \pi_1(D)$. Let $x\colon X \to D$ be given. A primary operation $\Phi_{X,x}\colon H^n(X; x^*\Gamma) \to H^k(X; x^*\Gamma')$ is represented by a map from $(L_\Gamma(G, n), D)$ to $(L_\phi(H, k), K(\pi, 1))$ which extends the classifying map for Γ'. This map determines a unique map from $L_\Gamma(G, n)$ to $L_{\Gamma'}(H, k)$ which is a map in $TD =$ the category of spaces under and over D (see [3]).

Higher order operations can be defined by towers of $K(\pi, 1)$-principal fibrations such as the following one:

$$\begin{array}{ccccccc} L_r & & & L_3 & & L_2 & & L_1 \\ \uparrow & & & \uparrow & & \uparrow & & \uparrow \\ E_r & \longrightarrow \cdots \longrightarrow & & E_3 & \longrightarrow & E_2 & \longrightarrow & L \end{array}$$

where $L = L_\Gamma(G, n)$, $L_r = L_\Psi(H, k)$, and $L_j = L_{\phi_j}(G_j, n_j)$, $\phi_j\colon \pi \to \operatorname{aut} G_j$, $1 \leqq j \leqq r - 1$, and $E_j \to L_j$ sends D to $K(\pi, 1)$. More specifically, Φ is defined by the following commutative diagram:

$$\begin{array}{ccc} H^n(X; x^*\Gamma) & \longrightarrow & H^k(X; x^*\Gamma') \\ \updownarrow & & \updownarrow \\ [X, L_\Gamma(G, n)]_D \longleftarrow & [X, E_r]_D & \longrightarrow [X, L_r]_K\,. \end{array}$$

The L's may be replaced by products over $K(\pi, 1)$ to give operations in several variables.

Suppose in the stable case that $f\colon D \to B$ is given and that all coefficient systems for the above tower are obtained by pullback from systems over B. Assume also that $f^*\colon H^*(B) \to H^*(D)$ is isomorphic for $i < N$ and for all coefficient systems involved. Then there is a B-operation Ψ such that if $X \xrightarrow{x} D \xrightarrow{f} B$ the operations $\Phi_{X,x}$ and $\Psi_{X,fx}$ are the same. This is proved by constructing a tower over B whose pullback is the given tower which defines the D-operation Φ.

The following lemma accounts for the "additive" in the main theorem.

LEMMA. *If $N+1 \leqq n_j < 2N$ for all n_j then the operations defined above are additive relations.*

The proof will be given for secondary operations of one variable. The argument for higher order operations of several variables is quite similar (see [3] for the constant coefficient case).

First, we recall some facts from [3]. Let D be any space. Top D is defined to be the category whose objects are triples $(W, \check{w}, \hat{w})$ with $\check{w}: D \to W$, $\hat{w}: W \to D$, and $\hat{w}\check{w} =$ identity. The morphisms are continuous functions $f: W \to W'$ such that $\hat{w}f = \check{w}'$ and $f\check{w} = \check{w}'$. Homotopy is defined in the natural way. The path and loop functors are defined as in § 2. There are cone and suspension functors also. All of the basic properties of Top (pt) remain valid for Top D (Top (pt) is just the category of pointed spaces and maps). If $W, W' \in \text{Top}\, D$, let $\langle W, W' \rangle$ be the set of homotopy classes of maps from W to W'. If W' is a [double] loop space in Top D then $\langle W, W' \rangle$ is a natural [Abelian] group. If $f: W \to W'$ is a loop map then f_* is a homomorphism.

Proof of the lemma for primary operations. Let α be a primary operation. By definition α is represented by a map in Top D (also denoted by α) $\alpha: L_\Gamma(G, n) \to L_{\Gamma'}(H, n')$. If $n' < 2n$ (as in the present theorem) then it can be shown that α is a loop map in Top D so the operation is a homomorphism.

The map $\bar{P}W \to W$ defined by $l \to l(1)$ is a fibration in Top D. Any fibration induced from it by a map to W (classifying map) is called a principal fibration. If the classifying map is a loop map then the total space of the principal fibration is a loop space.

Consider the following two diagrams

$$\begin{array}{ccc} E & \xrightarrow{\alpha_2} & L_2 \\ \downarrow p & & \\ L_\Gamma & \xrightarrow{\alpha_1} & L_1 \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\beta_2} & L_2' \\ \downarrow p & & \\ L_\Gamma & \xrightarrow{\beta_1} & L_1' \end{array} .$$

(1) (2)

$L_\Gamma = L_\Gamma(G, n)$, $L_i = L_{\phi(i)}(G_i, n_i)$, $L_i' = L_{\Gamma(i)}(G_i, n_i) = D \times_K L_i$. Γ, Γ_i are three given local coefficient systems on D classified by $\phi: \pi \to \text{Aut}\, G$, $\phi_i: \pi \to \text{Aut}\, G_i$. $\beta_i = (\mu, \alpha_i)$ where u is the map to D obtained from $E \to L_\Gamma \to D$. The important point is that $p: E \to L_\Gamma$ is the principal fibration induced by β_1 in Top D. This can be proved directly from the definitions.

Proof of the lemma for secondary operations. Consider the following commutative diagram.

$$
\begin{array}{ccccc}
H^n(X; x^*\Gamma) & & \xrightarrow{\qquad\qquad \Delta \qquad\qquad} & & H^{n(2)}(X: x^*\Gamma_2) \\
\updownarrow & & & & \updownarrow \\
[X, L_\Gamma(G, n)]_D & \xleftarrow{p_*} & [X, E]_D & \xrightarrow{\alpha(2)_*} & [X, L_2]_K \\
\updownarrow & & \updownarrow & & \updownarrow \\
[X, L_\Gamma(G, n)]_D & \xleftarrow{p_*} & [X, E]_D & \xrightarrow{\beta(2)_*} & [X, L_2]_D \\
\updownarrow & & \updownarrow & & \updownarrow \\
\langle XVD, L_\Gamma(G, n)\rangle & \xleftarrow{p_*} & \langle XVD, E\rangle & \xrightarrow{\beta(2)_*} & \langle XVD, L_2\rangle
\end{array} \quad .
$$

Δ is the secondary operation defined by line 2. The diagram shows that $\Delta = \beta(2)_* p_*^{-1}$ via line 4. Line 4 is obtained from Top D. $p: E \rightarrow L_\Gamma$ is a loop map and L_2 is a loop space in Top D. It can be shown that the dimension hypothesis of the lemma implies that $\beta(2)$ is a loop map in Top D. Hence p_* and $\beta(2)_*$ are both homomorphisms and Δ is an additive relation.

If $D = K(\pi, 1) = K$ the operations discussed above are closely related to those defined by Siegel in [7]. The difference is as follows. Let local coefficient systems Γ and Γ' on $K(\pi, 1)$ be represented by

$$
L_\phi(G, n) \underset{\check{l}}{\overset{\hat{l}}{\rightleftarrows}} K(\pi, 1) \qquad L_\psi(H, k) \underset{\check{m}}{\overset{\hat{m}}{\rightleftarrows}} K(\pi, 1) \ .
$$

Then in the present paper a primary operation is represented by a map $f: L_\phi(G, n) \rightarrow L_\psi(H, k)$ which is a K-map, i.e., $\hat{m}f = \hat{l}$ and $f\check{l} = \check{m}$. In Siegel's paper [7], a primary operation may be represented by a map which is merely a map over K, i.e., $\hat{m}f = \hat{l}$. The narrower definition of the present paper excludes the "characteristic operations" of [1, 7]. The advantage of the present approach is that all of the primary operations in the stable range are homomorphisms and the higher operations are additive relations. Presumably, the broader definition of [7] is more suitable for non-stable obstruction theory.

4. **Proof of the theorem.** Consider the following diagram where the left hand column is obtained from the right hand column by pullback. Let $s_j: E_i \rightarrow E_{i,j}$ be the natural section for the projection $p_j: E_{i,j} \rightarrow E_i$. Let k'_j be the composition

$$
E_{i,j} \longrightarrow E_j \xrightarrow{k_i} C_j \ .
$$

$$\begin{array}{ccccc} E_{i,i} & \longrightarrow & E_i & \xrightarrow{k_i} & C_i \\ \downarrow & & \downarrow & & \\ E_{i,i-1} & \longrightarrow & E_{i-1} & \longrightarrow & C_{i-1} \\ \vdots & & & & \\ E_{i,2} & \longrightarrow & E_2 & \longrightarrow & C_2 \\ \downarrow & & \downarrow & & \\ E_i & \longrightarrow & B & \xrightarrow{k_1} & C_1 \end{array} .$$

Claim. (1) $j \leqq i - 1$ implies $k'_j s_j \sim 0$ (over K).
(2) $k'_i = \bar{k}_i + u$, $k_i p_i = \bar{k}_i$, $\bar{k}_i s_i = k_i$, and $u s_i \sim 0$ (over K).

Proof of claim. (1) is clear since $k'_j s_j = E_i \to E_j \to C_j \sim 0$ (over K). Write $\bar{E}$ for $E_{i,i}$ so

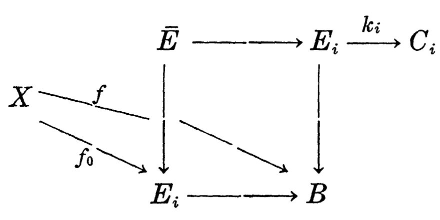
$$(\bar{E}, E_i) \supset \bar{E} \overset{s=s_i}{\supset} E_i \quad ps = 1 , \qquad \xrightarrow[p = p_i]{}$$

Hence

$$0 \to H^*(\bar{E}, E_i) \xrightarrow{g^*} H^*(\bar{E}) \underset{p^*}{\overset{s^*}{\rightleftarrows}} H^*(E_i) \to 0$$

where H^* is $[\ , C_i]_K$. $k'_i s_i = k_i$, that is, $s^* k'_i = k_i$. Hence $k'_i = \bar{k}_i + u$ where $\bar{k}_i = p^* k_i$ and $u = g^* u'$ and u and $\bar{k}_i$ are unique. Hence $\bar{k}_i s_i = k_i ps = k_i$. Also $k_i = k'_i = \bar{k}_i s_i + u s_i = k_i + u s_i$ implying $u s_i = 0$. This proves the claim.

Consider the following diagram:



where f_0 is a fixed lifting of f to E_i. Let $k = k_i$, $k' = k'_i$, and $E = E_i$. Then $O_i(f) = \{\hat{f}^* k \mid \hat{f}: X \to E \text{ lifts } f\} = \{\bar{f}^* k' \mid \bar{f}: X \to \bar{E} \text{ lifts } f_0\}$ (since the liftings of f and f_0 are in one-to-one correspondence)

$$\begin{aligned} &= \{\bar{f}^* \bar{k} + \bar{f}^* u\} = \bar{f}^* \bar{k} + \{\bar{f}^* u \mid \bar{f} \text{ lifts } f_0\} \\ &= \bar{f}^* \bar{k} + \{\bar{f}^* u \mid \bar{f} \text{ lifts } f'_0: X \to E_{i,2} \text{ and } f'_0 \text{ lifts } f_0\} \\ &= \bar{f}^* \bar{k} + \Phi_{X,f_0}[X, \bar{\Omega} C_1]_K . \end{aligned}$$

Use: $E_{i,2} = E_i \times_K \bar{P} C_1$, $[X, E_{i,2}]_{E_i} = [X, \bar{P} C_1]_K = [X, \bar{\Omega} C_1]$. Φ is the E_i-

operation defined by $E_{i,2} \leftarrow \bar{E} \rightarrow C_i$. If C_1 is the product of $L(G, n)'s$ over K then $E_{i,2}$ is a corresponding product over E_i of $L_{E_i}(G, n-1)'s$.

This shows that $O_i(f)$ is a coset of the image of the stable E-operation Φ defined by $E_{i,2} \leftarrow \bar{E} \rightarrow C_i$. The hypotheses of the theorem and the remarks in the third paragraph of § 3 guarantee that there is a stable B-operation Ψ with the same domain and image. This completes the proof.

REFERENCES

1. S. Gitler, *Cohomology operations with local coefficients*, Amer. J. Math. **85** (1963), 156–188.
2. M. Mahowald, *The third and fourth obstructions in sphere bundles* (1962?), unpublished.
3. J. F. McClendon, *Higher order twisted cohomology operations*, Invent. Math. **7** (1969), 183–214.
4. ———, *Obstruction theory in fiber spaces* (Submitted for publication)
5. J. P. Meyer, *The indeterminacy of certain cohomology operations* (1968?), unpublished.
6. P. Olum, *On mapping into spaces in which certain homotopy groups vanish*, Ann. of Math. **57** (1953), 561–573.
7. J. Siegel, *Higher order cohomology operations in local coefficient theory*, Amer. J. Math. **89** (1967), 909–931.
8. N. Steenrod, *The Topology of Fiber Bundles*, Princeton University Press, Princeton, 1951.

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