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MATRIX RINGS OF FINITE DEGREE OF NILPOTENCY

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The degree of nilpotency of a ring R is defined to be the supremum of the orders of nilpotency of its nilpotent elements and it is denoted by $\nu(R)$. We consider the degree of nilpotency of the ring of $m \times m$ matrices R_m over a ring R. We obtain given results concerning the degrees $\nu(R_m)$ for distinct m's, in the case R has no nonzero two-sided annihilators. It is shown that if $\nu(R_m) = m$ for some m, and if R' is a ring containing R as an ideal such that R' has no nonzero two-sided annihilators of R, then $\nu(R'_m) = m$. An application of this result is given.

R will always be a nonzero associative ring. If $a \in R$ is nilpotent, we denote its order of nilpotency by $\nu(a) = \min\{k \mid a^k = 0\}$, and if a is not nilpotent we put: $\nu(a) = 0$. The degree of nilpotency $\nu(R)$ of R is defined by

$$\nu(R) = \sup_{a \in R} \nu(a) .$$

If R is a ring without nonzero nilpotent elements then $\nu(R) = \nu(0) = 1$, and we shall soon see that the ring R_m of $m \times m$ matrices over R satisfies $\nu(R_m) \geqq m$ (Lemma 1).

There exist rings R satisfying $\nu(R_m) > m$ and in [3] was shown that such an R may even be a (noncommutative) integral domain. The object of this paper is to deal with rings R which satisfy $\nu(R_m) = m$ for some m. We denote this condition by $\mathfrak{N}_m$. First we shall consider the degree of nilpotency of matrix rings over rings without nonzero two-sided annihilators. Then we give some conditions equivalent to $\mathfrak{N}_m$. Our main result is: If a nonzero ideal in an integral domain R satisfies $\mathfrak{N}_m$ then R itself satisfies $\mathfrak{N}_m$. This implication resembles the following one: If a nonzero ideal in an integral domain R is embeddable in a field then R itself is embeddable in a field [1]. This result together with other results obtained in [4], lead us to the conjecture: "The conditions $\mathfrak{N}_m$, $m = 1, 2, \cdots$, are sufficient for embedding an integral domain in a field.

Our result is applied to prove that a ring which has no nonzero two-sided annihilators and satisfies $\mathfrak{N}_m$ is embeddable in a ring with an identity which satisfies $\mathfrak{N}_m$.

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2. Rings without nonzero two-sided annihilators. The following notations will be used later.

If $a \in R$ then we denote by $aEij$ the matrix with a in its (i, j) position and 0 elsewhere.

If $A = (a_{ij}) \in R_m$ and r is an integer $\geqq 1$, we denote the (i, j) entry of A^r by $a_{ij}^{(r)}$. Since $A^r A^s = A^{r+s}$ we have:

(1) $$\sum_{k=1}^{m} a_{ik}^{(r)} a_{kj}^{(s)} = a_{ij}^{(r+s)} .$$

LEMMA 1. *If R is not nilpotent then $\nu(R_m) \geqq m$ for each $m \geqq 1$.*

Proof. The result is trivial for $m = 1$, so let $m \geqq 2$. Since $R^{m-1} \neq 0$, there exist $a_1, \cdots, a_{m-1} \in R$ such that $a_1 \cdots a_{m-1} \neq 0$. Hence the matrix $A = \sum_{i=1}^{m-1} a_i E_{i,i+1}$ satisfies $A^{m-1} = a_1 \cdots a_{m-1} E_{1m} \neq 0$ and $A^m = 0$. Thus, $\nu(R_m) \geqq \nu(A) = m$.

COROLLARY. *For rings R without nonzero nilpotent elements, the condition $\mathfrak{N}_m$ is inherited by* (nonzero) *subrings.*

Indeed, if R' is a subring of R then $\nu(R'_m) \geqq m$ since R' is not nilpotent. If R satisfies $\mathfrak{N}_m$ then since R'_m is a subring of R_m we have $\nu(R'_m) \leqq \nu(R_m) = m$.

If S is a nonempty subset of R, we denote its right (left) annihilator in R by $r_R(S)(l_R(S))$. Clearly $r_R(S) \cap l_R(S)$ is the set of two-sided annihilators of S in R.

Note that if R is a (nonzero) ring such that $r_R(R) \cap l_R(R) = \{0\}$ then R is not nilpotent.

The proof of our next result is similar to that of [4, Lemma 9].

LEMMA 2. *If $r_R(R) \cap l_R(R) = \{0\}$ and $A \in R_m$ is nilpotent of order h, then there exist a matrix $B \in R_{m+1}$ which is nilpotent of order $h + 1$.*

Proof. If $h = 1$ then $A = 0$ and the result is trivial. If $h \geqq 2$ then $A^{h-1} \neq 0$ and there exist p and q, $1 \leqq p, q, \leqq m$, such that $a_{pq}^{(h-1)} \neq 0$. Since $r_R(R) \cap l_R(R) = \{0\}$, there exists an element $b \in R$ such that either $ba_{pq}^{(h-1)} \neq 0$ or $a_{pq}^{(h-1)} b \neq 0$. Assume that we have $a_{pq}^{(h-1)} b \neq 0$ (the other case is treated similarly). Let A_1 be the matrix of R_{m+1} obtained from A by adjoining a row and a column of zeros and let $B = A_1 + bE_{q,m+1}$. The powers of B are given by

$$B^k = A_1^k + \sum_{i=1}^{m} a_{iq}^{(k-1)} bE_{i,m+1},\ k \geqq 2 .$$

Since $A_1^h = 0$ and $a_{pq}^{(h-1)} b \neq 0$ we obtain $B^h \neq 0$ and $B^{h+1} = 0$.

This immediately yields:

THEOREM 3. *Let R be a ring such that $r_R(R) \cap l_R(R) = \{0\}$. If $\nu(R_m) \geqq h$ then $\nu(R_{m+r}) \geqq h + r$ for each $r \geqq 1$, and if $\nu(R_m) \leqq h$ then $\nu(R_{m-r}) \leqq h - r$ for each $r = 1, 2, \cdots, m - 1$.*

THEOREM 4. *If $r_R(R) \cap l_R(R) = \{0\}$ and R satisfies $\mathfrak{N}_m$ for some m, then it also satisfies $\mathfrak{N}_k$ for $k = 1, 2, \cdots, m-1$. In particular it follows that R has no nonzero nilpotent elements.*

3. Conditions equivalent to $\mathfrak{N}_m$.

THEOREM 5. *Let m be a fixed integer > 1. The following conditions are equivalent for rings R without nonzero nilpotent elements.*

(i) $\mathfrak{N}_m$: $\nu(R_m) = m$

(ii) *For all $C \in R_m$, $C^{m+1} = 0$ implies $C^m = 0$.*

(iii) *For all $A, B \in R_m$, $(AB)^m = 0$ implies $(BA)^m = 0$.*

Proof. It is clear that (i) implies (ii). If (ii) holds and $(AB)^m = 0$ then $(BA)^{m+1} = B(AB)^m A = 0$, hence $(BA)^m = 0$ and (iii) holds.

Assume (iii) holds and we proceed to prove (i). Since R has no nonzero nilpotent elements $r_R(R) = l_R(R) = 0$, so $\nu(R_m) \geqq m$. Let $C = (c_{ij}) \in R_m$, we have to prove that $\nu(C) \leqq m$. Assume $\nu(C) = h > m$ and let $c_{pq}^{(h-1)} \neq 0$. We define two matrices $A = (a_{ij}) \in R_m$ and $B = (b_{ij}) \in R_m$ as follows:

$$a_{ij} = \begin{cases} c_{pj}^{(i)}, & i = 1, \cdots, m-1 \\ c_{pj}^{(h-1)}, & i = m \end{cases}, \qquad j = 1, \cdots, m .$$

$$b_{ij} = c_{iq}^{(h-j)}, \qquad i, j = 1, \cdots, m .$$

Using (1) we obtain for $j = 1, \cdots, m$

$$\sum_{k=1}^{m} a_{ik} b_{kj} = c_{pq}^{(h+i-j)}, \qquad i = 1, \cdots, m-1 .$$

$$\sum_{k=1}^{m} a_{mk} b_{kj} = c_{pq}^{(h+h-1-j)} .$$

Since $C^h = 0$, it follows that $C^{h+r} = 0$ and $c_{pq}^{(h+r)} = 0$ for each $r \geqq 0$. Hence the (i, j) entry of AB is 0 for $i \geqq j$, and it is $c_{pq}^{(h-1)}$ for $j = i + 1, i = 1, \cdots, m-1$. This implies that $(AB)^{m-1} = (c_{pq}^{(h-1)})^{m-1} E_{1m}$ and $(AB)^m = 0$. Since (iii) holds we have $(BA)^m = 0$. But

$$(BA)^m = B(AB)^{m-1} A$$

and its (i, j) entry is $b_{i1}(c_{pq}^{(h-1)})^{m-1} a_{mj} = 0$. Taking $i = p$ and $j = q$ we obtain $(c_{pq}^{(h-1)})^{m+1} = 0$ and since R has no nonzero nilpotent elements, it follows that $c_{pq}^{(h-1)} = 0$, a contradiction. Hence $h \leqq m$ and R satisfies (i).

4. The main result. If $T \neq 0$ is an ideal in R and T as a ring satisfies $\mathfrak{N}_m$, then it does not follow that R satisfies $\mathfrak{N}_m$, even if R has no nonzero nilpotent elements. Indeed, R may be a direct sum of T and a ring R' such that $\nu(R'_m) > m$ and it is possible to choose

T and R' without nonzero nilpotent elements. Clearly, here the two-sided annihilator of T in R is not 0. On the other hand we have:

THEOREM 6. *If T is an ideal in R such that $r_R(T) \cap l_R(T) = \{0\}$ and $\nu(T_m) = m$, then $\nu(R_m) = m$.*

Proof. We have $r_T(T) \cap l_T(T) \subseteqq r_R(T) \cap l_R(T) = 0$ and $\nu(T_m) = m$, hence it follows by Theorem 4 that T has no nonzero nilpotent elements. Since R_m contains T_m we have $\nu(R_m) \geqq m$. Let $C \in R_m$, we have to prove that $\nu(C) \leqq m$. As in the proof of Theorem 5, assume $\nu(C) = h > m$ and $c_{pq}^{(h-1)} \neq 0$. Construct the same matrices A and B and take arbitrary elements $a, b \in T$. Then $A_1 = aA$ and $B_1 = Bb$ belong to T_m. We have $A_1B_1 = a(AB)b$, hence the (i, j) entry of A_1B_1 is 0 for $i \geqq j$ and it is $ac_{pq}^{(h-1)}b$ for $j = i + 1, i = 1, \cdots, m - 1$. From this it follows that $(A_1B_1)^{m-1} = (ac_{pq}^{(h-1)}b)^{m-1}E_{1m}$ and $(A_1B_1)^m = 0$. Since $A_1, B_1 \in T_m$ and $\nu(T_m) = m$ it follows that $(B_1A_1)^m = 0$. As in the proof of Theorem 5 we obtain that the (p, q) entry of $B_1(A_1B_1)^{m-1}A_1 = 0$ is

$$c_{pq}^{(h-1)}b(ac_{pq}^{(h-1)}b)^{m-1}ac_{pq}^{(h-1)} = 0 .$$

This implies that

$$(bac_{pq}^{(h-1)})^{m+1} = 0, (ac_{pq}^{(h-1)}b)^{m+1} = 0, (c_{pq}^{(h-1)}ba)^{m+1} = 0 .$$

Since T has no nonzero nilpotent elements it follows that

$$bac_{pq}^{(h-1)} = 0, ac_{pq}^{(h-1)}b = 0, c_{pq}^{(h-1)}ba = 0 .$$

This is true for all $a, b \in T$, hence $ac_{pq}^{(h-1)} \in r_T(T) \cap l_T(T) = \{0\}$ and $c_{pq}^{(h-1)}b \in r_T(T) \cap l_T(T) = \{0\}$ and this implies that $c_{pq}^{(h-1)} \in r_R(T) \cap l_R(T) = \{0\}$; a contradiction. Hence $h \leqq m$ and $\nu(R_m) = m$.

If R is an integral domain and T a nonzero ideal in R, then it is clear that $r_R(T) = l_R(T) = \{0\}$, hence we obtain our main result which is:

THEOREM 7. *If R is an integral domain and $T \neq 0$ an ideal in R which satisfies $\mathfrak{N}_m$, then R also satisfies $\mathfrak{N}_m$.*

5. **Embedding.** Let R be a ring without nonzero nilpotent elements. Embed R in a ring R' with 1 in the usual way [2, p. 86]: $R' = R + I$, $R \cap I = 0$, where I is the ring of integers. R is an ideal in R' and since $r_R(R) = l_R(R) = \{0\}$ it follows that $r_{R'}(R) \cap R = l_{R'}(R) \cap R = \{0\}$. Thus, R is embeddable in $R'/r_{R'}(R) = R''$. One shows easily that $r_{R'}(R) = l_{R'}(R)$. If we identify R with its image in R'' we obtain that R is an ideal in R'' and $r_{R''}(R) = \{0\}$. Hence by Theorem 6 we obtain:

THEOREM 8. *If R is a ring without nonzero nilpotent elements and satisfies $\mathfrak{N}_m$, then R is embeddable in a ring with* 1 *which satisfies* $\mathfrak{N}_m$.

If R is an integral domain then the ring R'' obtained above is also an integral domain. Thus, we have:

COROLLARY. *If R is an integral domain which satisfies $\mathfrak{N}_m$ then R is embeddable in an integral domain with* 1 *which satisfies* $\mathfrak{N}_m$.

Note that this result enables us to simplify the proof in [4, Theorem 7] taking $t = 1$.

Now, if R is a ring with 1 and satisfies $\mathfrak{N}_m$ then R has no nonzero nilpotent elements since $r_R(R) = \{0\}$. Let C be the center of R and assume that the nonzero elements of C are regular in R. Thus, we may embed R in the ring $R' = \{ac^{-1} \mid a \in R, 0 \neq c \in C\}$ whose center is the quotient field of the commutative integral domain C. If $B = (b_{ij}) \in R'_m$ then it is possible to write its entries with a common denominator: $b_{ij} = a_{ij}c^{-1}, a_{ij} \in R, 0 \neq c \in C, 1 \leqq i, j \leqq m$. Let $A = (a_{ij}) \in R_m$ then $Bc = A$. If B is nilpotent then A is also nilpotent and since R satisfies $\mathfrak{N}_m$ we have $A^m = 0$. It follows that $B^m c^m = (Bc)^m = 0$ and so $B^m = 0$ since c^m is a unit in R'. We have proved:

THEOREM 9. *If R is a ring with* 1 *which satisfies $\mathfrak{N}_m$ and all the elements of its center C are regular, then R is embeddable in a central K-algebra which satisfies $\mathfrak{N}_m$, K the field of fractions of C.*

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